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PART I

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LOGIC

IN THREE BOOKS

HOUGHT OF INVESTIGATIO

AND OF KNOWLEDGE

BY

HERMANN LOTZE

ENGLISH TRANSLATION, EDITED BY

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EDITOR'S PREFACE.

SINCE the present Translation of Professor Lotze's 'System der Philosophie' was begun, both the author himself, who cordially welcomed the undertaking, and Professor Green, who first definitely proposed it, have been removed by death. These two distinguished men, however different in method and style of thought, had some fundamental tendencies in common; and it may be of interest to Professor Lotze's admirers in this country to know that Professor Green not only executed an important part of the Translation[1], but intended to take upon himself the task of revising and editing the whole[2], which was not entrusted to the present Editor till after Professor Green's death.

The Translation of the Logic has been throughout adapted to the second edition. But the Author's intended revision of the Metaphysic was not carried out, and the projected Part III of the 'System der Philosophie' was never written. What the Author made known of his intentions in these respects is mentioned in the Prefaces to the Metaphysic.

The translation of Part I, the Logic, has been executed by several hands; the whole of Book I by Mr. R. L. Nettleship, Fellow of Balliol College, Oxford; Book II,

[1] See Preface to the 'Metaphysic.'

[2] He said to the present Editor: 'The time which one spent on such a book as that (the "Metaphysic") would not be wasted as regards one's own work.'

chapters i–v (inclusive), by Mr. F. H. Peters, Fellow of University College, Oxford (with the exception of the 'Note on the Logical Calculus,' which was translated by the Editor); chapters vi–ix (inclusive), by Mr. F. C. Conybeare, Fellow of University College; and chapter x by the Editor; and the whole of Book III by Mr. R. G. Tatton, Fellow of Balliol College.

The Editor has revised the whole translation, and is responsible in all cases for the rendering finally adopted. He has to thank Mr. J. C. Wilson, Fellow of Oriel College, Oxford, for the most cordial and ample assistance in dealing with the numerous passages in which mathematical knowledge was required. It is believed that the translation of these passages will, owing mainly to his help, be found on the whole correct and intelligible.

The Table of Contents was furnished by the several translators for their respective portions. It should be observed that the original Table of Contents supplies a few headings (in Book I only), besides those of the chapters. These are distinguished from the headings supplied by the translator by being printed in italics. The Index was added by the Editor.

No endeavour has been made to introduce uniformity of style into the different portions of the translation. But in the case of a few important technical terms it has been thought advisable to introduce renderings as nearly uniform as the context would allow. Unavoidable variations in the translation of a German word, or ambiguities in the employment of an English one, are pointed out, to some extent, in the Notes and Index; and in all cases references are freely given to any passages that explain the precise point of the Author's choice of words. It is hoped that by

this means the reader may be assisted to master the somewhat subtle distinctions which govern the Author's usage, without the aid of a Glossary, which could indicate them but roughly. Still Professor Wallace's observations on the meaning of some German terms, prefixed to his translation of Hegel's shorter Logic, will be found useful by many readers.

In the case of two of the sections which treat of mathematical questions (234 and 237) the Editor found himself in a perplexity which could have been removed if the Author had been still living. The reasoning of sect. 234 seemed more than doubtful; while the Author himself had requested the suppression of sect. 237 as 'wholly erroneous,' regretting that he had put forward such 'nonsense,' and explaining that he had been 'misled by the error of a text-book.'

This unqualified condemnation seemed on consideration hardly to apply to sect. 237, and to be such as might have been intended for sect. 234; but as the Author mentioned not only the number of sect. 237, but the pages on which it stands, the hypothesis of a mere clerical error is almost excluded. It is nevertheless conceivable that there may have been some misapprehension; and therefore it has been thought advisable not to withdraw sect. 237 entirely, but to print it as an Appendix.

In preparing the second English edition of the 'System der Philosophie' no important alterations have been made in the translation, although a few verbal corrections have been found necessary.

AUTHOR'S PREFACE.

THOUGH I venture to describe the present work as the first part of a System of Philosophy, I hope that this designation will not be supposed to indicate the same pretensions which it was wont to herald in times gone by. It is obvious that I can propose to myself nothing more than to set forth the entirety of my personal convictions in a systematic form; such a form as will enable the reader to judge not only to what degree they are consistent with themselves, but also how far they are capable of serving to unite the isolated provinces of our certain knowledge, in spite of the great gaps that lie between them, into a coherent view of the world bearing the character of completeness. In the present volume, which begins my exposition, I have been guided, as I shall be in the others, by this purpose.

In the First Book, although entirely rewritten, I have followed in essentials the line of thought of my short work on Logic of 1843, which has long been out of print. I have not seen reason to depart from this line, to which my own interest in the exposition of Logic is as much confined now as it was then. Now, as then, I consider it useless labour to attempt extensions and improvements of the formal part of Logic, within the limits of the general character which in fact and of necessity attaches to it; but whatever in it appeared worth knowing, if only as belonging in a certain sense to the history of culture, I have to the best of my belief conveyed without omission, and have taken pains to do so as simply as possible.

The Second Book needs no preface; it is quite free

from the bonds of system, and simply puts together whatever I thought useful. The selection of matter might be different in many parts, a great deal might be added, and a great deal, it will be thought, might be spared. The reader should regard it as an open market, where he may simply pass by the goods he does not want.

The original purpose of the Third Book has not been carried out. It was meant to treat of the subjects with which it does in fact deal, on the method of a historico-critical exposition of systematic logical views—the views which have appeared both in Germany and in several other countries in a variety of forms that demand a high degree of interest and appreciation. But it became clear on making the attempt that such a task could not be achieved within the limits of the present treatise,—not, that is to say, with the thoroughness due to the valuable works in question. Another opportunity may possibly be found for it; but in the meantime I was induced by the failure of this plan to dispense for the present with all reference to doctrines which are not my own, and to put forward nothing but what either is common property, or belongs to my own individual mode of viewing the subject. I trust that the whole of my doctrine is not merely of this latter kind!

GÖTTINGEN: *June* 10, 1874.

The present (2nd) edition contains a number of improvements in detail, and a single addition of some length, the 'Note on the Logical Calculus,' p. 208 (E. Tr. vol. i. p. 275). I may remark with reference to p. 222 (E. Tr. vol. i. p. 297) that Jevons speaks of Potassium. Perhaps the reader can conjecture why I have preferred to speak of Sodium.

GÖTTINGEN: *September* 6, 1880.

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BOOK I.

OF THOUGHT (PURE LOGIC).

INTRODUCTION.

I. At almost every moment of our waking life our senses are giving rise to various ideas, simultaneous or immediately successive. Among these ideas there are many which have a right thus to meet in our consciousness, because in the reality from which they spring their occasioning causes always accompany or follow one another; others are found together in us merely because, within the external world to whose influence we are accessible, their causes were as a fact simultaneous though not so inwardly connected as to ensure their similar combination in every recurring instance. This mixture of *coherent* with merely *coincident* ideas is repeated, according to a law which we derive from self-observation, by the current of memory. As soon as any idea is revivified in consciousness, it reawakens the others which have once accompanied or succeeded it, whether the previous connexion was due to a coherence in the matter of the ideas, or to the mere simultaneity of otherwise unconnected irritants. It is upon the first fact, the recovery of what is coherent, that our hope of arriving at knowledge is based: the second, the ease with which coincident elements hang together and push one another into conscious-

ness, is the source of error, beginning with that distraction which hinders our thoughts from following the connexion of things.

II. The ever-changing whole of processes which results from this peculiarity of our psychical life is what we call the *current of ideas*. If it were in our power to observe this whole with omniscience, we should discover in every instance of it, in the sober course of waking thought, in the dreams of sleep, in the delirium of disease, a necessary connexion between its members. The application of universal laws, which hold good of all souls alike, to the particular conditions which are found to vary in each single instance, would exhibit the course of these inner processes in the light of an inevitable result. If we knew the permanent characteristics of a single particular soul, if we had a view of the form and content of its whole current of ideas up to the present time, then, the moment it had produced a first and a second idea on occasion of external irritants, we should be able to predict on the basis of those universal laws what its third and fourth idea in the next moment must be. But in any other soul, whose nature, past history, and present condition were different, the same first and second idea, developed at this moment by a similar external irritant, would lead with a similar necessity in the next moment to an entirely different continuation. An investigation of the subject would therefore have to recognise that any given current of ideas was necessary for that particular soul and under those particular conditions; but it would not discover any mode of connexion between ideas which was universally valid for all souls. And just because, under their respective conditions, every such series of ideas hangs together by the same necessity and law as every other, there would be no ground for making any such distinction of value as that between truth and untruth, which would place one group in opposition to all the rest.

III. Universal validity and truth are the two prerogatives which even ordinary language ascribes and confines to those connexions of ideas which *thought* alone is supposed to establish. Truth is familiarly defined as the agreement of ideas and their combinations with their object and its relations. There may be objections to this form of expression, which this is not the place to consider; but it will be innocuous if we modify it and say, that connexions of ideas are true when they follow such relations in the matter of the ideas as are identical for all consciousness, and not such merely empirical coincidence of impressions as takes one form in one consciousness, another in another. Now our ideas are excited in the first instance by external influences, and this leads us to regard thought as a reaction of the mind upon the material supplied by those influences and by the results of their interaction already referred to. The thinking mind is not content to receive and acquiesce in its ideas as they were originally combined by casual coincidence or as they are recombined in the memory: it sifts them, and where they have come together merely in this way it does away with their coexistence: but where there is a material coherence between them, it not only leaves them together but combines them anew, this time however in a form which adds to the fact of their reconnexion a consciousness of the ground of their coherence.

IV. I will connect the indispensable explanation of what I have just said with the elucidation of some obvious objections. It is not without a purpose, which I admit, that while I have represented the rest of the current of our ideas as a series of events, which happen in us and to us according to universal laws of our nature, I have represented thought as an activity which our mind exercises. There have been persons who doubted whether this opposition has any real significance, either in itself or in relation to thought; whether everything that we are in the habit of calling activity is not rather one amongst the events which

simply take place in us. So wide a question does not of course admit of being decided here: if therefore I hold to the significance of the opposition, and expressly describe thought as an activity, this must be regarded as a presupposition which awaits proof elsewhere, but is at present open to dispute. It is necessary for the connexion of the whole to which I wish this view of thought to serve as an introduction; and it seems to me to be permissible, because, while it will determine decisively the general colour of my exposition, it will not alter unnaturally the internal relations of its subject-matter.

V. It is more profitable to meet another form of the same objection, which allows the general validity of the opposition in question, but holds that there is no occasion to apply it here. The connexion of the coherent (it is said), that is to say, Truth, is brought about in the same way, only not quite so soon, as the erroneous conjunctions of the casually coincident. The course of things itself ensures that those events which are inwardly connected exercise their combined effect upon us with incomparably greater frequency than those which have no inward bond, but are variously thrown together by chance. Owing to this more frequent repetition the connexion of what is coherent becomes fixed in us, while that of the merely coincident is loosened and disturbed by its want of uniformity. In this way the separation of the coherent from the incoherent, which we thought it necessary to ascribe to a special reaction of the mind, is effected by the current of ideas itself; and thus brutes, like men, acquire the mass of well-grounded information which regulates the daily life of both. It would be superfluous to point out that this account is perfectly correct if it purport to be no more than a history of the acquisition just mentioned; but I think it can be shown that this acquisition is just what neither characterises nor exhausts the specific work of thought.

VI. There is a common opinion which reserves the

faculty of thought to man and denies it to brutes. Without seriously deciding for or against this view, I will use it for the convenience of my explanation. In the soul of a brute, which on this theory would be confined to a mere current of ideas, the first impression of a tree in leaf would only produce a collective image; there would be no power or even impulse to seek for any special coherence between its parts. Winter strips the tree of its leaves, and on a second observation the brute finds only a part of the former collective image, which tries to reproduce the idea of the rest, but is hindered by the present appearance. When the return of summer restores the old state of things, the renewed image of the whole tree in leaf may not, it is true, have the simple unquestioning unity of the first observation; the recollection of the second intervenes, and separates it into the part which remained and the part which changed. I do not think we can say what precisely would take place in the soul of the brute under these circumstances; but even if we ascribe to it the additional faculty of comparing and surveying the current of its ideas and expressing the result, the expression could not say more than the fact that two observations were at one time together, at another not. Now it is true that the man, when he gives the name of leafy and leafless tree to the same observed objects, is only expressing the same facts; but the apprehension of the facts, which is indicated by these habitual forms of speech, involves a mental operation of quite a different kind. The name of the tree, to which he adds and from which he takes away the descriptive epithet, signifies to him, not merely a permanent as opposed to a changeable part in his observation, but the thing in its dependence on itself and in opposition to its property. The effect of bringing the tree and its leaves under this point of view is, that the relationship of thing and property appears as the justification both for separating and for combining these ideas, and thus the fact of their coexistence or non-coexistence in our

consciousness is referred to the real condition upon which their coherence or non-coherence at the moment depends.

The same consideration may be extended to other instances. In the soul of the dog the renewed sight of the raised stick recalls the idea of his previous pain: the man, when he makes the judgment, 'the blow hurts,' does not merely express the fact of connexion between the two occurrences, but justifies it. For in representing the blow as the subject from which the pain proceeds, he clearly exhibits the general relationship of cause and effect as the ground, not of the mere coexistence of the two ideas in us, but of their right and obligation to follow one another. Lastly, the expectation of pain in the dog may be accompanied by the recollection that by running away, to which he was led before by an involuntary instinct, the pain is diminished; and this fresh conjunction of ideas will doubtless make him repeat the salutary operation as surely as if he reflected and concluded that threatening blows are prevented by distance, that a blow threatens him, and that therefore he must run away. But the man who in a similar or more serious case actually frames such a conclusion, performs an entirely different mental operation; in expressing a universal truth in the major premiss, and bringing a particular instance under it in the minor, he not merely repeats the fact of that salutary connexion between ideas and expectations by which the brute is affected, but he justifies it by an appeal to the dependence of the particular upon its universal.

VII. These examples, which embrace the familiar forms of thought, concept, judgment, and syllogism, will I think have made sufficiently clear what is the surplus of work performed by thought over and above the mere current of ideas; it always consists in adding to the reproduction or severance of a connexion in ideas the accessory notion of a ground for their coherence or non-coherence. The value of this work remains entirely the same, whatever opinion we

may hold of its genesis: if we preferred to regard it, not as the outcome of a special activity, but only as a finer product of the mere current of ideas operating under favorable circumstances, we should then confine the name of thought to that particular stage of development in the current at which it gives birth to this new achievement. The peculiarity of thought, then, which will govern the whole of our subsequent exposition, lies, not in the mere correspondence of our apprehension with fact, but in the production of those accessory and justificatory notions which condition the form of our apprehension. We do not deny that, apart from thought, the mere current of ideas in the brute gives rise to many useful combinations of impressions, correct expectations, seasonable reactions; on the contrary, we admit that much even of what the man calls his thought is really nothing but the play of mutually productive ideas. And yet perhaps there is still some difference here. The sudden inspirations which enable us to make a decision in a moment, the rapid survey which arranges a complicated material in almost less time than would seem sufficient for the bare observation of its parts, the invention of the artist which remains unconscious of the grounds by which it is impelled, all these seem to us to be effects, not of a current of ideas which has not yet become thought, but of abbreviated thought. In the cases where these surprising operations are successful, they are so because mature thought has already in other cases developed into full-grown habits those accessory notions, which bring the impressions under universal principles of coherence; and this, like all other accomplishments which have acquired the ease of a second nature, has behind it a forgotten time of laborious practice.

VIII. In the examples which I have employed, the accessory notions, by which we justify the connexions of ideas, obviously coincided with certain presuppositions about the connexions of the real with which we cannot

dispense. Without the opposition of things and their properties by which the whole matter of perception is articulated, without the assumption of a succession of effects from causes, and without the determining power of the universal over the particular, we could have no apprehension whatever of the reality which surrounds us. From this point of view, then, it seems a self-evident proposition that the forms of thought and the accessory notions which give them vitality are immediate copies of the universal forms of being and its connexions, and this *real* validity of thought and its operations has, in fact, been frequently maintained. The opposite view to this, which as its exact counterpart we might expect to find, has never been put forward so unreservedly. To the unprejudiced mind it is too natural to regard thought as a means of comprehending the real, and any interest in the scientific investigation of its processes is too dependent upon this presupposition for any one to assert the merely *formal* validity of all logical activity in the sense of denying *all* relation between it and the nature of being. Those, therefore, who have regarded the forms and laws of thought as being primarily peculiar results of our mental organisation, have not wholly excluded their correspondence with the essence of things; they have only denied the off-hand view which would make them immediate copies of the forms of being.

IX. In regard to this much debated question an introduction can only take up a provisional position. We shall certainly be right to confine our attention at starting to what is already clear, and to leave for a later stage the decision of uncertainties. Let us then go no further than the natural presupposition which regards thought as a *means* to knowledge. Now a tool must fulfil two conditions, it must fit the thing and it must fit the hand. It must fit the thing; that is, it must be so constructed as to approach, reach, and get hold of, the objects which it is to work upon,

and find in them a point from which to operate; this requirement is satisfied in the case of thought if we admit that its forms and laws are no mere singularities of our mental organisation, but that, taken as they are, they show a constant and regular adaptation to reality. If, again, a tool is to fit the hand, it must have such other structural properties as make it easy to grasp, hold, and move, having regard to the power, attitude, and position of the person who is to use it; and in the case of thought this second indispensable requirement limits the scope of the previous admission. Only a mind which stood at the centre of the real world, not outside individual things but penetrating them with its presence, could command such a view of reality as left nothing to look for, and was therefore the perfect image of it in its own being and activity. But the human mind, with which alone we are here concerned, does not thus stand at the centre of things, but has a modest position somewhere in the extreme ramifications of reality. Compelled, as it is, to collect its knowledge piece-meal by experiences which relate immediately to only a small fragment of the whole, and thence to advance cautiously to the apprehension of what lies beyond its horizon, it has probably to make a number of circuits, which are immaterial to the truth which it is seeking, but to itself in the search are indispensable. However much, then, we may presuppose an original reference of the forms of thought to that nature of things which is the goal of knowledge, we must be prepared to find in them many elements which do not directly reproduce the actual reality to the knowledge of which they are to lead us: indeed there is always the possibility that a very large part of our efforts of thought may only be like a scaffolding, which does not belong to the permanent form of the building which it helped to raise, but on the contrary must be taken down again to allow the full view of its result. It is enough to have thus raised a preliminary expectation, with which we wish our subject

to be met; any more definite decision as to the limits which separate the formal validity of our thought from its real significance must await the further course of our enquiries.

X. I have purposely avoided postponing those enquiries by discussions which seem to me to encumber unjustifiably the approach to logic. What particular tone of mind is required for successful thinking, how the attention is to be kept up, distraction avoided, torpidity stimulated, precipitation checked, all these are questions which no more belong to the field of logic than do enquiries about the origin of our sense-impressions and the conditions under which consciousness in general and conscious activity is possible. We may presuppose the existence of all these things, of perceptions, ideas, and their connexion according to the laws of a psychical mechanism, but logic only begins with the conviction that the matter cannot end here; the conviction, that between the combinations of ideas, however they may have originated, there is a difference of truth and untruth, and that there are forms to which these combinations *ought* to answer and laws which they *ought* to obey. It is true that we may attempt by a psychological investigation to explain the origin of this authoritative consciousness itself; but the only standard by which the correctness of our results could be measured would be one set up by the very consciousness to be investigated. The first thing, then, that has to be ascertained is, *what* the contents of this authoritative conviction are; the history of its growth can only have the second place, and even then must conform to requirements of its own imposing.

XI. Having now said all that seemed necessary by way of introduction to my exposition, I will add a preliminary survey of its order. The examples which we have hitherto employed lead naturally to a first principal part, which, under the name of pure or formal logic, is devoted to thought in general and those universal forms and principles

of thought which hold good everywhere, both in judging of reality and in weighing possibility, irrespective of any difference in the objects. We have only to mention concept, judgment, syllogism, to see how naturally these forms exhibit themselves as different stages of one and the same activity; and in treating of pure logic I shall endeavour to emphasise this thread of connexion somewhat more strongly than is usually done. The various forms of thought will be arranged in an ascending series, in which each higher member attempts to make good a defect in the preceding one, due to its failure to satisfy, in regard to its own particular problem, the general impulse of thought to reduce coincidence to coherence. This series will advance from the simplest formation of single impressions to the conception of the universal order in which this general impulse would lead us, if it were possible, to comprehend the world.

XII. Pure logic itself will show and explain that the forms of concept, judgment, and syllogism are to be considered primarily as *ideal* forms, which give to the matter of our ideas, if we succeed in arranging it under them, its true logical setting. But the different peculiarities of different objects offer resistance to this arrangement; it is not clear of itself what sum of matter has a claim to form a determinate concept and be opposed to another, or which predicate belongs universally to which subject, or how the universal law for the arrangement of a manifold material is to be discovered. Applied logic is concerned with those methods of *investigation* which obviate these defects. It considers hindrances and the devices by which they may be overcome; and it must therefore sacrifice the love of systematisation to considerations of utility, and select what the experience of science has so far shown to be important and fruitful. The boundlessness of the field of observation unfortunately makes it impossible to exhibit as completely as could be wished this most brilliant part of logic, which

the inventive genius of modern times has made peculiarly its own.

XIII. The third part will be devoted to *knowledge*, that is, to the question which our introduction touched without answering, the question how far the most complete structure of thought which all the means of pure and applied logic enable us to rear, can claim to be an adequate account of that which we seem compelled to assume as the object and occasion of our ideas. The currency in ordinary minds of this opposition between the object of our knowledge and our knowledge of that object makes me employ it without hesitation to describe in a preliminary way the subject of this third section; it may be left to the section itself to disclose the difficulties which this apparently simple antithesis involves, and to determine accordingly the more precise limits of the problems with which it has to deal.

CHAPTER I.

The Theory of the Concept.

A. *The formation of impressions into ideas.*

1. It is in relations within a manifold that the operations of thought usually show themselves to us, and we might therefore expect to have to look for the most original of its acts in some simplest form of connexion between two ideas. A slight reflexion, however, suggests to us to go a step further back. It is easy to make a heap out of nothing but round stones, if it is indifferent how they lie; but if a structure of regular shape is to be built, the stones must be already so formed that their surfaces will fit firmly together. We must expect the same in the case before us. As mere internal movements, the states which follow external irritants may exist side by side in us without further preparation, and act upon each other as the general laws of our psychical life allow or enjoin. But if they are to admit of combination in the definite form of a *thought*, they each require some previous shaping to make them into logical building-stones and to convert them from *impressions* into *ideas*. Nothing is really more familiar to us than this first operation of thought; the only reason why we usually overlook it is that in the language which we inherit it is already carried out, and it seems therefore to belong to the self-evident presuppositions of thought, not to its own specific work.

2. That which takes place in us immediately under the influence of an external stimulus, the sensation or the feeling, is in itself nothing but a state of our consciousness, a mood of ourselves. We do not always succeed in naming, and so making communicable to others, the manner in which we are thus affected; sometimes the formless interjection, the exclamation, is the only way we can find, though with no certainty of being understood, to give sound to what cannot be said. But in the more favorable cases, where we have succeeded in creating a name, what exactly is it which this creation effects and indicates? It is just what we are here looking for, the conversion of an *impression* into an *idea*. As soon as we give the name of green or red to the different movements which waves of light produce through our eyes, we have separated something before unseparated, our sensitive act from the sensible matter to which it refers. This matter we now present to ourselves, no longer as a condition which we undergo, but as a something which has its being and its meaning in itself, and which continues to be what it is and to mean what it means whether we are conscious of it or not. It is easy to see here the necessary beginning of that activity which we above appropriated to thought as such: it has not yet got so far as converting coexistence into coherence, it has first to perform the previous task of investing each single impression with an independent validity, without which the later opposition of their real coherence to mere coexistence could not be made in any intelligible sense.

3. We may describe this first operation of thought as the beginning of an *objectification* of the subjective; and I take advantage of this expression to guard against a misunderstanding and so illustrate the simple meaning of what I have said above. It is not objectivity in the sense of some sort of real existence which would subsist though nobody had the thought of it, that, by the logical act of creating a

name, is accorded to the subject-matter to which that act gives rise. The true meaning of the first act of thought is best exemplified by those languages which have maintained the use of the article. The article, which had everywhere originally the value of a demonstrative pronoun, marks the word which it accompanies as the name of something to which we point; and what we point to is something which admits of being observed by another person as well as by ourselves. This can be done most easily with things which have an actual position in space between the speakers; but developed language makes an object of any other matter of thought in the same way. Such objectivity, therefore (which in these cases also is indicated by the article), does not entirely coincide with the reality which belongs to things as such; it is only the fact of their claiming such a reality, on the ground of the distinctive peculiarity of their real nature, which language has met and expressed in their names. When we speak of[1] 'the tooth-ache,' 'the day,' 'the franchise,' we do not imply that they could exist if there were no person to feel, to see, to enjoy them, respectively. Still less when we talk of 'the adverb' or 'the conjunction,' do we mean to indicate by the article that the subject-matter described by these words has any sort of existence outside thought. We only mean that certain special forms of resistance and tension, which we feel in the course of our ideas, are not only peculiarities of our own state and inseparable from it, but that they depend upon relations inherent in the matter of various ideas, which every one who thinks those ideas will find in them just as we do.

The logical objectification, then, which the creation of a name implies, does not give an external reality to the

[1] [The instances in the text are *der Schmerz*, *die Helligkeit*, *die Freiheit*, but none of the equivalents are used in the required sense with the article in English. The same applies to the instances in the following sentence, *das Zwar*, *das Aber*, *das Dennoch*.]

matter named; the common world, in which others are expected to recognise what we point to, is, speaking generally, only the world of thought; what we do here is to ascribe to it the first trace of an existence of its own and an inward order which is the same for all thinking beings and independent of them: it is quite indifferent whether certain parts of this world of thought indicate something which has besides an independent reality outside the thinking minds, or whether all that it contains exists only in the thoughts of those who think it, but with equal validity for them all.

4. But the objectification of the matter so first constituted is not the whole of this first act of thought; consciousness cannot simply present the matter to itself, it can only do so by giving it a definite position; it cannot simply distinguish it from an emotional mood of its own, without accrediting it with some other sort of existence instead of that which belonged to it as such a mood. The meaning of this requirement (for I admit that my expression of it is not immediately clear) is most simply shown by the way in which language actually satisfies it. It is only in the interjection, which is not a name of definite content, that language retains the formlessness which belongs to it as the mere expression of excitement; the rest of its stock of words is articulated in the definite forms of substantives, adjectives, verbs, and the familiar *parts of speech* in general. And it is hardly necessary to insist that the various characters thus impressed by language upon its material are the indispensable condition of the later operations of thought; it is obvious that neither the combination of marks into the concept, of concepts into the judgment, or of judgments into the syllogism would be possible, if the matter of every idea were equally formless or apprehended in the same form, if some of them were not substantival and did not express fixed and independent points of attachment for others which are adjectival, or if

others again were not verbal, exhibiting the fluid relations which serve to bring one thing into connexion with another. I do not think it advisable to separate this particular conformation of the matter of ideas, as a second act of thought, from the first act, to which we ascribed its objectification; I prefer to comprise the primary activity of thought in a single operation, which may be indifferently represented as that of giving to the matter of ideas one of these logical forms by making it objective for consciousness, or as that of making it objective by giving it one of these forms.

5. The three parts of speech which I have noticed remind us inevitably of three concepts which are indispensable for our judgment of reality. It is impossible to have even an expressible idea of the world of perception, without thinking of things in it as fixed points which serve to support a number of dependent properties, and are connected together by the changing play of events. If metaphysic is the investigation, not of the thinkable in general, but of the real or that which is to be recognised as such, these concepts of thing, property, and event are metaphysical concepts; not perhaps such as metaphysic would finally allow to stand without modification, but certainly such as at its outset purport to represent immediately the proper essence and articulation of what is.

It would seem at first sight that the logical forms of substantivity, adjectivity, and verbality coincide with these concepts: but a second view shows the same difference between the two series as that which separated the logical objectification of an idea from external reality. Nothing passes with us for a thing or a substance which has not reality outside us and permanence in time, producing changes in something else and capable of undergoing changes itself; but we apprehend as substantives not only things but their properties; as substantives we speak of 'change,' 'occurrence,' even of 'nothing,' and so in in-

numerable cases of that which has no existence at all or none except in dependence on something else. Thus the substantival form invests its content, relatively to the future predicates to which it is to serve as subject, with only the same priority and independence as belong to a thing in contrast with its properties, conditions, and effects, but by no means with that concrete and independent reality and activity which place a thing above a mere object of thought.

Verbs, again, express most frequently an event which as a fact takes place in time; but when we say that things 'are,' or 'are at rest,' or that one 'conditions' or 'equals' another, it is clear that the verbal form too does not universally give to its content the meaning of an event, but only finds it there usually. In order to conceive fully the sense of such verbs as we have just instanced, we have to connect several distinct contents together by a movement of thought, and this movement though it implies time for its execution, is, as regards its meaning and intention, quite independent of time. In a word, the general sense of the verbal form is not an event, but a relation between several related points; and this relation may just as well occur between contents which are out of time and coexist only in thought, as between those which belong to reality and are accessible to temporal change.

Lastly, while it is true that radical adjectives, such as 'blue' and 'sweet,' express primarily what appears to our first apprehension as a real property of things, every developed language knows words like 'doubtful,' 'parallel,' 'allowable,' which, as the least reflexion shows, can no longer mean in the same simple sense as the former a property attaching to actual things; they are abbreviated and condensed expressions of the result of all sorts of relations, and it is only for purposes of thought that we represent the contents of such adjectives as related to those of substantives in the way in which we imagine an

attribute to be related to its subject. Speaking generally, then, the logical import of the parts of speech is only a shadow of the import of these metaphysical concepts: it only repeats the formal characteristics which the latter assert of the real; but by not confining their application to the concrete external reality, it loses that part of their significance which they only possess in that application.

6. Lastly, if we found in the forms of the parts of speech the most original activity of thought, we must also understand how to distinguish this from its linguistic expression. Now that man has come to use the language of sounds for the communication of his thoughts, that activity is, it is true, most clearly manifested in the forms of the parts of speech; but in itself it is not inseparably bound up with the existence of language. The development of which the ideas of the deaf and dumb are capable, though guided in the first instance by those who can speak, is enough to show that the internal work of logic is independent of the possibility of linguistic expression. That work consists merely in the fact that we accompany the content of one idea with the thought of its comparative independence, while we think of another as requiring support, and of a third as a connecting link which neither subsists on its own account nor rests upon something else but mediates between two others. No one doubts the extremely effective support which language gives to the development of thought by making the formations and transformations of ideas vividly objective to consciousness by means of sharply defined sounds and their regular changes; still, if some other mode of communication were natural to man instead of the language of sounds, the same logical associations would find in it a corresponding expression though of a different kind. And if in some languages the poverty of forms does not always allow these associations to take shape, cannot, for instance, distinguish between substantival and verbal construction, yet there is no doubt that the mind

of those who speak them maintains the logical distinctions while forming ideas which are vocally undistinguished. Wherever there is this inward articulation, there is thought; where it is wanting, there is no thought. For this reason music is not thinking; for however manifold and delicately gradated are the relations of its tones, it never brings them into the position of substantive to verb, or into dependence such as that of an adjective on its noun or a genitive on the nominative by which it is governed.

7. In mentioning hitherto only three out of a greater number of parts of speech, the three without which the simplest logical enunciation would be impossible, I do not wish to deny the value of the others. But the road which we have to traverse is too long to allow us to make further circuits into the attractive field of philological enquiry, circuits which, considering how thought has just been said to be independent of its mode of expression, must for our purpose remain circuits. The articulation and usage of language do not fully cover the work of thought. We shall find later that they frequently do not express the complete structure of the thought; and then we have for the purposes of logic to supplement what is said by what was meant. On the other hand language possesses technical elements which do not depend, or only depend with various degrees of remoteness, upon characteristics essential to logic: in such cases we should not be justified in distinguishing a different logical operation of thought for every grammatical or syntactical difference of form presented by language. There are not only interjections, but particles too, which, like the tone of the voice, hardly indicate more in ordinary usage than the interest which the speaker feels in what he is saying, and contribute nothing to its substantial logical meaning. When language introduces the distinction of gender into all substantives and adjectives, it follows an æsthetic fancy which has no interest for logic; when on the other hand it determines the gender of the

adjective by that of its substantive, this consistency in an arbitrarily adopted custom points back to a logical relationship which we shall become acquainted with. When in the inflexions of the verb it distinguishes the person speaking from the person spoken to and the third person not present, it emphasises an extremely important fact in a way which is indispensable for the living use of speech, and yet there is no corresponding distinction in logic proper. It is nothing but the same reason which justifies grammar in considering pronouns as a specific class of parts of speech: logically, the personal pronouns must be reckoned entirely among substantives, with which in formal position they are identical; the possessive and demonstrative we have no ground for separating from adjectives; the relative we should regard as the most specifically technical element in language, serving only the need of methodical communication, and based on no other logical relation than its counterpart the demonstrative. Numerals are treated by grammar as distinct parts of speech; in the actual usage of language they are equivalent to adjectives, and that logically they belong to the latter we cannot doubt, when we remember that logically the form of adjectivity belongs to all characteristics of a subject-matter which are not self-dependent, and not only to those which attach to it in the sense of properties. Adverbs, lastly, stand in precisely the same relation to the meaning of verbs as adjectives to that of substantives, so that logic would have no occasion to consider them as a distinct part of speech or a peculiar form of the content of thought.

Thus there would only remain prepositions and conjunctions to put forward such a claim, and of them I think we must admit that, however they may be derived linguistically, they form an indispensable element in the world of our ideas. They cannot be derived from the concept of relation, with which at first they seem to be connected: whenever two members are connected by a relation, there

is involved the thought of a certain position which those members occupy within the relation itself, and this position need not be the same for both; on the contrary, it is generally different, the one embracing, containing, and conditioning the other. Now it will be found upon trial to be impossible to express this difference of value between the related points, without which the relation has no meaning, in a merely verbal form: somewhere or other we shall need a preposition, a conjunction, or at least one of the various case-forms in which many languages express some of these accessory notions still more shortly. In what linguistic form they appear, is of course quite indifferent to logic; just as we oppose the nominative, as that which conditions, sometimes to the genitive, as that which is conditioned, sometimes in a different sense to the accusative, so, if language had produced or preserved a still greater wealth of cases, all prepositions would be superfluous, as all conjunctions would be if there were a similar variety of moods. This would make no change in the logical needs of thought; in one way or another, the meanings of substantives, adjectives, and verbs would have to be supplemented by a number of ideas, indicating, either, like prepositions, the position of two supposedly simple objects in a simple relation, or, like conjunctions, the comparative position and value of two relations or judgments.

8. If we glance at the developed structure of the world of our thoughts and ask what the conditions are upon which its construction depends, the objectification of impressions and their concomitant formation in the sense of the parts of speech must always appear as the most indispensable, and in that sense the first, of all operations of thought. It is certain that without it the framing of sentences, simple or complex, through which we express the work and results of our thinking, would have been quite impossible. But we must not be taken to mean that the logical spirit, at the beginning of its intellectual

work, before it ventured a step further, performed this, the first of its necessary operations, on the entire matter of its ideas once for all. The infinitude of possible impressions, of which every moment may bring a new one, would be enough to make such a task impracticable: it is made still more impracticable by the fact that in working up the matter that is given to it thought is constantly producing new matter, and has to bring this again into the same logical forms of which, as applied to a simpler matter, it is the result. Thus it is that every developed language possesses, in the form of simple substantives, adjectives, or verbs, numerous ideas which could not have been framed, and cannot be fully understood, without manifold intellectual operations of a higher kind, without employing judgments and syllogisms, and even without presupposing systematic scientific investigation.

This obvious reflexion has given rise to the assertion, that in logic the theory of judgment at least must precede the treatment of concepts, with which it is only an old tradition to begin the subject. I consider this to be an over-hasty assertion, due partly to a confusion of the end of pure with that of applied logic, partly to a general misconception of the difference between thought and the mere current of ideas. For if those judgments, out of which the concept is said to result, are to be really judgments, they themselves can consist of nothing but combinations of ideas which are no longer mere impressions; every such idea must have undergone at least the simple formation mentioned above; the greater part of them, as experiment would show, will already practically possess that higher logical form to which the very theory in question gives the name of concept. The element of truth in this proposed innovation reduces itself to the very simple thought, that in order to frame complex and manifold concepts, more especially in order to fix the limits within which it is worth while and justifiable to treat them as

wholes and distinguish them from others, a great deal of preparatory intellectual work is necessary; but that this preparatory work itself may be possible, it must have been preceded by the conformation of simpler concepts out of which its own subsidiary judgments are framed. Without doubt, then, pure logic must place the form of the concept before that of the judgment: it remains for applied logic to tell us how, in framing determinate concepts, judgments consisting of simpler concepts may be turned to account. A proposal to reverse this order can only commend itself to those who regard thinking in general as merely the interaction of impressions excited in us from without, and overlook the reacting energy which makes itself felt at every point in the current of ideas, separating the merely coincident, combining the coherent, and thus already giving form to the individual elements of future thoughts.

B. *Position*[1], *Distinction, and Comparison of the Matter of Simple Ideas.*

9. If we recognise in these first formative acts the specific contribution which the operative energy of thought makes to the whole of our intellectual world, we are easily led to the view that the logical spirit has certain ready-made modes of apprehension with which it meets the impressions as they come; and this again raises the question, how it contrives to bring the matter of each impression under that particular form which is appropriate to it. But such a view is inadmissible, and such a question therefore has no point, or at any rate leads to an answer different from that which it expects. Thought does not stand fronting the impressions as they arrive with a bundle of logical forms in its hand, uncertain which form can be fitted to which impression, and therefore needing some special expedient

[1] ['Position,' as the equivalent of *Setzung*, is here used in the active sense in which it occurs, e. g. in 'composition.']

to discover how to pair them properly. It is the relations themselves, already subsisting between impressions when we become conscious of them, by which the action of thought, which is never anything but reaction, is attracted; and this action consists merely in interpreting relations, which we find existing between our passive impressions, into aspects of the matter of the impressions. It is not therefore the assignment of the proper form of each matter, which requires any special device of thought: in another point of view, however, this arrangement of a manifold matter in logical forms does involve a second intellectual operation; for no matter can have a *name* made for it unless it has been thought of as identical with itself, as different from others, and as comparable with others.

10. This second operation of thought, like the first, is one which inherited language has already carried out for all those who speak it; like the first therefore it is easily overlooked, and not reckoned as part of the work of the mind. But logical science, expressly devoted to the self-evident, must not treat a part of its subject as a still more self-evident presupposition which may be excluded from the proper objects of its consideration. Still, the first at any rate of the three heads under which we expressed this new operation of thought does not need a detailed explanation. It is at once obvious how every name, 'sweet' or 'warm,' 'air' or 'light,' 'tremble' or 'shine,' gathers up the matter which it indicates in some sort of coherent unity with a meaning of its own; it is not only (though it is most emphatically) matter in the substantival form that is thus lifted into unity with itself by the prefixed article; the same indicative force resides, under a different form, in the infinitive of the verb, and even when language has no distinctive expression for it, this accessory notion of singling out and giving position to the matter indicated accompanies every form of word. It may be doubted whether the process which we would understand by *giving*

position is not already contained in the objectification which we represented as converting the passive impression into an idea; and it is true that we can neither have an idea without thus giving position to its content, nor give it position in any intelligible sense without objectifying it. Practically therefore it is a really inseparable operation which we are considering from different sides; before, we contrasted the presented idea to which we are related as presenting, with the impression by which we are simply affected; now, when the multiplicity of the matter presented begins to excite our attention, we lay stress upon the unity and independence in virtue of which the matter thus singled out by attention is what it is and differs from everything else.

11. By the last words I wished to convey clearly the close connexion in which the affirmative position given to a content stands with the negative *exclusion* of all others. The connexion is so close, that the terms which we are obliged to employ to express the simple sense of the first are only made perfectly clear by adding the accessory notion of the second. We can only explain what we mean by the unity of position given to a content by emphasising its difference from others, and saying, not only, it is what it is, but also, it is not what others are. The affirmation and the negation are one inseparable thought, and accompany in inseparable union every one of our ideas, even when we do not expressly attend to the others which are tacitly negated. But the accessory notion thus amalgamated with our ideas only determines the logical setting which we give to their content; it does not produce that content in the first instance. It cannot be said that we have the idea of red as red only when we distinguish it from blue or sweet, and only by so distinguishing it, and again the idea of blue as blue only by a similar opposition to red. There could be no conceivable occasion for attempting such a distinction, nor any possibility of succeeding in the attempt, unless

there were first a clear consciousness of what each of the two opposites is in itself. Without doubt the peculiar impression which we experience under the influence of red light will be entirely the same before we have had our first experience of blue light as it will be afterwards; the possibility of comparison and distinction which the latter experience gives may indeed, at any rate in a matter more complex than these simple colours, draw the attention to parts of the impressions which had been previously overlooked, and so make both of them more complete; but even in this case, which is quite outside our present consideration, the new element is not discovered by the distinction, but by the immediate sensation of which the comparison was merely the occasion. It is always affirmative position therefore which makes negative distinction possible, while it is never the case that the act of distinction gives rise to the matter distinguished. Only our accessory notions about the matter of our ideas, only its logical setting, gains in definiteness by adding to the affirmation of itself the negation of others; and even this gain would seem to me small if it went no further, and were not supplemented by that third operation of positive comparison, which, in the above account of this second act of thought, was mentioned last.

12. I will introduce the consideration of this third operation, which I regard as the most essential part of the logical work to be here explained, by recalling a familiar fact which is commonly used to support other conclusions. Words never denote impressions as they can be experienced; we can only experience or actually perceive a particular shade of red, a specific kind of sweetness, a definite degree of warmth, not the universal red, sweet, and warm of language. The universalisation which in these and all similar cases the matter of sensation has undergone, is commonly regarded as an unavoidable inexactness of language, perhaps even of the thought which language

serves to express. Unable or not accustomed to make a definite name for every single impression, language (it is supposed) blurs the slight differences between them, and retains only what is immediately experienced in sensation as common to them all: by this reduction of its means of expression to a moderate number it certainly makes the communication of ideas possible, but diminishes proportionately the exactness of that which has to be communicated. I do not think that this view does full justice to the significance of the fact.

13. First of all, to regard the universalisation in question as a sort of falsification of the impressions is to pass too lightly over the very remarkable circumstance, that in a number of different impressions there *is* something common which can be thought apart from their differences. This is by no means such a matter of course that the opposite is out of the question; on the contrary, it is quite conceivable that every one of our impressions should be as incomparably different from every other as sweet actually is from warm, yellow from soft. The fact that the thinkable world itself is so constituted that this is not the case, is one which it is worth while to take into consideration. Nor again can I regard the want of exactness, which the application of the universal terms of language undoubtedly gives to the communication of ideas, as sheer loss. Moreover, when perfect exactitude is felt to be important, the shortcomings of these simplest products of rudimentary thought can always be supplemented by its more advanced activity: science has long taught us to measure every degree of heat, and in case of necessity would find out how to measure every gradation of redness or sweetness.

But the way in which language and natural thought operative in language solve the same problem, seems to me to be logically very significant. For when, instead of attaching a particular name to every single colour of which we have actual sensation, we give the privilege of names of

their own to blue, red, yellow, and a few others, and then intercalate the other individual sensations between them as bluish red or reddish yellow, this is not merely a shift for approximating to an unattainable exactitude; rather, as it seems to me, it expresses the conviction that only these few colours are really fixed points deserving names of their own, while the rest must be characterised by approximate expressions because they are themselves only approximations to these fixed points, or connecting links between them. If we really had particular and mutually independent names for every single shade of blue, and our ideas answered to this form of expression, we should have achieved in a one-sided way the separation of each from every other, but we should have overlooked completely the positive relations which subsist between them all. If on the contrary we speak of bright blue, dark blue, black blue, we arrange this manifold in a series or a network of series, and in each series a third member results from a second by intensification of the same sensible change in a common element as that which gave rise to the second out of the first. It must be already perfectly clear that a presentative activity which did not involve this comparison of the diverse, but was confined to the bare separation of each from each, would not offer to the later operations of thought adequate grounds for contrasting two ideas, as in some way or other cohering, with two others as not cohering. We therefore apprehend this second act of thought, of which we are here speaking, not merely as that of giving simple position to *a* or *b*, not merely as that of simply distinguishing every *a* from every *b*, but also as that of determining the extent and peculiarity of the distinction, which is not everywhere the same in degree and kind, but is different between *b* and *c* and between *a* and *b*. I do not mean to say that every single idea, *a*, must be accompanied by the developed idea of all its relations to the infinite number of all other ideas; the general accessory notion, that every idea is enclosed on all sides in such a

network of relations, does indeed in our logical consciousness envelop every idea; but these relations are only followed out in each particular case so far as a special requirement suggests.

14. This comparison of the diverse clearly presupposes a common element to which in the several members of the series specific differences attach. Such a common element is usually considered by logic only in the form of a universal concept, and in this shape it is a product of more or less numerous acts of thought. It is therefore important to point out that this *first universal*, which we find here involved in the comparison of simple ideas, is of an essentially different kind; that it is the expression of an inward experience which thought has merely to recognise, and that just for this reason it is, as will be seen later, an indispensable presupposition of that other kind of universal which we shall meet with in the formation of concepts. We impart the universal concept of an animal or a geometrical figure to another person by directing him to execute a precisely definable series of intellectual operations, connecting, separating, or relating a number of simple ideas assumed to be known; when this logical work is completed, we suppose him to have before his mind the same object-matter which we wished to impart to him. But we cannot explain by the same means wherein the universal blue or the universal colour consists, which accompany our ideas of bright and dark blue or of red and yellow. We can indeed direct another person to think of all single colours or all shades of blue, and by eliminating their differences bring out what is common to his ideas in the two cases; but it is only in appearance a logical work which we are here prescribing; all that we really call upon him to do is to see for himself how he executes the task. How he is to set to work to discover whether there really is any common element in red and yellow, and how he is to contrive to separate it from the differences, this we cannot tell him; we

must simply trust to his having an immediate sensation, feeling, or experience of the connexion which exists between red and yellow, of the fact that they contain a common element; his logical work can consist only in the recognition and expression of this inward experience. This first universal, therefore, is no product of thought, but something which thought finds already in existence.

15. I will insert an observation here which with slight modification may be extended to all universals, but is most easily illustrated in this simplest instance, the first universal. That in which red and yellow agree and which makes them both colours cannot be separated from that which makes red red and yellow yellow, not separated, that is to say, so as to form the content of a third idea similar in kind and order to the two compared. It is always, as we know, only a single definite shade of colour, only a tone of definite height, strength, and quality, which is the object of sensation; and it is only these definite impressions which are so repeated in memory as to present substantial and perceptible images to consciousness. Universal ideas never have this perceptibility. If we try to apprehend the universal element of colour or tone, we shall always find that either we have before our perception a definite colour and a definite tone, only with the accessory notion that every other tone and colour has an equal right to serve as a perceptible instance of the ever imperceptible universal; or else our memory will produce a number of colours and tones in succession, with the same accessory notion that it is not these individuals that are really meant, but the common element in them which cannot as such be apprehended in perception. If therefore we understand by idea[1] (as ordinary usage certainly inclines us) the consciousness of something standing at rest before the mind, or a perception of something capable of being presented to it, the universal cannot claim to be called an idea. Words like 'colour' and 'tone' are

[1] [Vorstellung.]

in truth only short expressions of logical problems, whose solution cannot be compressed into the form of an idea. They are injunctions to our consciousness to present to itself and compare the ideas of individual tones and colours, but in the act of so comparing them to grasp the common element which our sensation testifies them to contain, but which cannot by any effort of thought be really detached from their differences and made the material of a new and equally perceptible idea.

16. Let us now direct our attention to the differences, which, within the first universal, separate the various instances of it. It is clear that what distinguishes one sensation of warmth from another, a gentler from a louder sound, bright from dark blue, is a more or a less of a common sensible element, which in itself, undetermined by any degree, is no object of perception. We shall find ourselves brought back to the same ground of distinction in all other ideas; it is only in giving an account of the universal, to which this quantitative comparison applies, that we meet with a difficulty, which after the above remarks is intelligible. The louder tone is no doubt distinguished from the gentler by a certain intensification, but so also is the higher from the lower; yet it is only in the former case that we feel able to express directly, by the term 'strength,' the common element which undergoes this change; in the latter we express it by the metaphor of height. Red and yellow seem to be still more essentially different and underivable one from the other by increase or decrease of a common element; only the intermediate colours, reddish yellow or yellowish red, are intelligible to us as mixtures containing more or less of one or the other. Nevertheless no one denies that one of the fundamental colours is more nearly related to a second than to a third, red to yellow than to green; and these grades of resemblance cannot be conceived without a more or a less of some common element, which we are conscious of in passing

from one member of the series to the next and from this to the third. To determine in each particular case what this common element consists in, to decide whether a number of ideas are separated merely by differences in degree of one simple universal, or by differences in value of several mutually determined ones, and whether accordingly the ideas are to be grouped in a linear series or plane-wise or in still higher forms, these are all attractive objects of enquiry, but they are not objects of logic. For logic it is enough to know that some generally applicable and primarily quantitative determination is the indispensable means for distinguishing between the particular instances of a universal. And even this determination is something which it is not the work of logic to produce, but only to find, recognise, and develop. A judgment, '*a* is stronger than *b*,' is indeed, as a judgment, a logical piece of work; but that which it expresses, the general fact that differences of degree do exist in the same matter, as well as the particular fact that the degree of *a* exceeds that of *b*, can only be experienced, felt, or recognised as part of our inward consciousness. By whatever artificial contrivances we may seek to increase scientifically the exactness of a measurement, everything must depend ultimately on the capacity to recognise two sensuous perceptions as like or as unlike, and not to be deceived as to which has the more and which the less.

17. If inward experience were confined to bringing out resemblances and differences in the various object-matters, thought would merely be called upon to arrange ideas in an unalterable system, like the musical scale, in which all tones have once for all their fixed and immoveable places. But logic has to do with thought, not as it would be under hypothetical conditions, but as it is. Now owing to the mechanism which controls the interaction of its inward states, all actual thought has necessarily more opportunities of stimulation than the above hypothesis would imply; the manifold matter of ideas is brought before us, not only in the

systematic order of its qualitative relationships, but in the rich variety of local and temporal combinations; and this fact, like the other, belongs to the material which serves thought in its further operations and must be given it to start with. The combinations of heterogeneous ideas produced in this way form the problems, in connexion with which the efforts of thought to reduce coexistence to coherence will subsequently have to be made. The homogeneous or similar ideas on the other hand give occasion to separate, to connect, and to count their repetitions; and to these ideas of unity and multiplicity those of greatness and smallness are added where the matter presented is continuously extended in space or in time. These three pairs of quantitative ideas (for we have already got those of more and less) comprise all the standards by which the individual instances of any universal are distinguished.

18. There are two things which I intentionally exclude from my consideration. Firstly, all enquiry into the psychological character of the growth and development of these quantitative ideas in our consciousness, into the order in which one of them may condition the origin of another, and into the different importance of perceptions of time and space in their formation. However attractive these questions may be, it would lengthen our way unnecessarily to answer them; logic is not concerned with the manner in which the elements utilised by thought come into existence, but with their value, when they have somehow or other come into existence, for the carrying out of intellectual operations. Now this point, which I conceive to have been unduly neglected, I wish to emphasize here, and shall subsequently keep in view, viz. that all ideas which are to be connected by thought must necessarily be accessible to one of the three quantitative determinations which have just been mentioned. The other thing which I exclude is the investigation of the consequences which may be drawn from these quantitative determinations as such: they

have long ago developed into the vast structure of mathematics, the complexity of which forbids any attempt to re-insert it in universal logic. It is necessary, however, to point out expressly that all calculation is a kind of thought, that the fundamental concepts and principles of mathematics have their systematic place in logic, and that we must retain the right at a later period, when occasion requires, to return without scruple upon the results which mathematics have been achieving, as an independently progressive branch of universal logic.

19. If we take a general survey of this second act of thought, in which I now include that of giving affirmative position to the object-matter, that of distinguishing it negatively from all others, and that of estimating by quantitative comparison its differences and resemblances, we may observe that the significance of this new logical operation is somewhat different from that of the first, by which impressions were shaped into ideas. In the former case there was a temptation (which, it is true, we resisted) to regard the forms of substantivity, adjectivity, and verbality as modes of apprehension which thought is ready to put in practice upon its object-matter before receiving any solicitation from it; but though we set aside this claim at once, it remains true that in those forms thought does not merely respond to and reproduce the actual current of ideas, but gives them the shape without which the logical spirit could not accept them. The independence which the substantival form gives to its matter, most obviously by means of the article, did not itself lie in the fact that this matter was a permanent link between changing groups of ideas; nor was the accessory notion of dependence expressed by the adjectival form present, as such, in the fact which stimulated the mind to characterise it by that form; so that we may continue to assert, in a certain sense, that in this first act thought dictates its own laws to its object-matter.

If, using an expression which we shall otherwise avoid, we represent this procedure as a proof of spontaneity, the second act of thought has the character of receptivity; it is a recognition of facts, and adds no other form to them except this recognition of their existence. Thought can make no difference where it finds none already in the matter of the impressions; the first universal, as we saw, can only be experienced in immediate sensation; as so experienced it can be named, but this is the only contribution which logic can make to the further fixing of its character; all quantitative determinations, to whatever extent thought may develop them by subsequent comparison, always come back to an immediate consciousness of certain characteristics given in the object-matter. I should wish this fact to be considered from two points of view. In the first place, logic is guilty of a certain carelessness in assuming at almost every moment in its later stages the comparability of ideas and the possibility of their subordination to a universal, without observing that that possibility, and the success of its own procedure in general, depends upon this original constitution and organisation of the whole world of ideas, a constitution which, though not necessary in thought, is all the more necessary to make thinking possible. For I must repeat that there is no inherent contradiction in supposing that every idea was incomparably different from every other; that in the absence of all qualitative comparability there was no standard of more or less; that the same idea never presented itself twice to perception; and that, as there was no repetition of the homogeneous, the ideas of larger and smaller also vanished. The fact that this is not the case, but that the world of ideas is organised as we have found it to be, must be emphasized as of the highest importance; but logic ought not in case of need to appeal to it incidentally as a self-evident truth derived no one knows whence. And this brings me to the other observation which I had to make. If thought is a reaction upon a

stimulus found in the current of ideas, a systematic survey of its functions will show clearly at certain points the influence exercised upon them by the thinkable world; as it is here the second member in the first triple series of operations, so at a later stage also it will be the second member of the following more highly developed group in which we shall see the peculiar dependence of thought upon the material to which it is directed. I do not however claim to do more by this preliminary indication than to throw a preliminary light over the system which I have followed in my exposition; the system itself can only find its justification in the advantages which in its successive stages it will be found to secure.

C. *The Formation of the Concept.*

20. To separate the merely coincident amongst the various ideas which are given to us, and to combine the coherent afresh by the accessory notion of a ground for their coherence, is the further task of thought. It will be useful, with a view to making its meaning clear, to review the different senses in which any combination of manifold elements occurs in our mental world. In the first place, no later intellectual activity is possible, unless the various ideas upon which it is to be exercised meet together in one and the same consciousness. The fulfilment of this condition is secured by the unity of the soul and the mechanism of memory, which, by bringing together impressions separated in time, makes their interaction possible. This union of the manifold may be called the synthesis of apprehension; it is not a logical act; it merely lumps the manifold together into a simultaneous possession of consciousness, without combining any two of its elements in a different order from any other two. Such an order comes in with the second form of connexion, the synthesis of perception, that is, with figures in space and succession in time, in which the individual impressions take up definite and non-equivalent

positions. This connexion also is supplied by the inward mechanism of consciousness without any action of thought, and however firmly defined and finely articulated it may be, it exhibits nothing but the fact of an external order, and reveals no ground of coherence justifying coexistence in that order. From the second stage I pass at once to the fourth, to a synthesis in which the last-mentioned requirement would be completely satisfied in regard to any given object-matter. In such a synthesis we should have before our mind, not the mere fact of manifold elements in order, but also the value which each element possessed in determining the coalescence of the whole. If what we thus apprehended were an object in real existence, we should see which were the prior, determining, and effective elements in it, in what order of dependence and development the others followed from them, or what end was to be regarded as their authoritative centre, involving in itself the simultaneous union or successive growth of them all: if, like the figures of geometry, it was something which had no reality out of our consciousness and no growth or development in time, we should here too attempt at any rate (though, as we shall see later, with limited success) to arrange the elements of the whole in a hierarchy in which those that conditioned others should take precedence of those that were conditioned, according to their stages of dependence. It is easy to see that a synthesis of this sort would be neither more nor less than the knowledge of the thing; as the goal of all intellectual effort, it lies as far above the province of logic as the first and second modes of connexion lay beneath it; it is in the space between that we must place the third and logical form of synthesis, the character of which has now to be examined.

21. When a person who has no special knowledge speaks of 'credit' or of 'banking,' we trace in these expressions his conviction that a number of businesses and institutions form a connected whole; but he would not be able to say where the nerve of the connexion lies, or what limits

separate the whole from that which does not belong to it. In this accessory notion, that the various elements are not merely there in a sort of heap, but form a whole of parts with self-imposed limits and a unity included by those limits, the general impulse of thought leaves its mark upon the given object in a formal way, without as yet attaining material fulfilment. If we pass our mental world in review, we are in this position as regards a very large part of its contents; indeed we shall be surprised to find that words of great significance betray this imperfect apprehension of their objects; for the more complex, important, and various any matter is, the more easily will persuasive impressions derived from repeated observations awaken the feeling of its individuality, completeness, and self-inclusiveness, without necessarily giving any real insight into its structure. Such words as 'nature,' 'life,' 'art,' 'knowledge,' 'animal,' and many others have no more significance than this in ordinary usage; they merely express the opinion that a certain quantity, usually not exactly definable, of individual objects, attributes, or events, which attach to one another, form somehow an inwardly connected whole, which can neither have any part taken away without being destroyed, nor admit any casual additions within the bounds of its unity. But how little the nature of this connexion is really known, appears from the failure of the attempt to describe the limits which include what belongs to the unity and exclude what does not. So long as the logical work of holding the manifold together has not gone further than this, I should hesitate to speak of 'concepts,' though I do not attach any value to the invention of a special technical term for such imperfect apprehension. Suppose we call it an imperfect or growing concept; then we shall not feel that we have got a perfect or fully developed concept, until the vague suggestion of some sort of whole has grown into the pervading thought that there is a definite ground for the co-existence of these particular attributes,

in this particular combination and to the exclusion of certain others, and that this ground is an adequate one.

22. The question now arises, how we get at this ground and condition. If we merely continued to observe a composite form *a b c d* in its isolation, we should never discover, however long we looked, which of its parts only coexist, which really cohere, and in what degree the existence of one depends upon that of another. But if we compare *a b c d* with other forms like it, that is, with such as we are led from it to observe, not by any special logical effort, but by the natural current of our ideas, and if we find that in *a b c d*, *a b c f*, *a b c g*, etc., a similar group *a b c* occurs with various dissimilar additions, we regard the latter as loose and separable appendages of the permanent stem *a b c*. Nor does the common group *a b c* contrast with the rest merely as the centre to which as a matter of fact they attach; on the general assumption that we have before us a whole of interdependent parts, this solid kernel becomes the expression of the constant rule which allows the accretion of the several accessory elements, and determines the manner in which it takes place. If we wish for practical purposes to ascertain in any creature, object, or arrangement, what is the line which divides what is inwardly coherent from casual accessions, we put the whole in motion, in the belief that the influence of change will show which parts hold firmly together while foreign admixtures fall away, and in what general and constant modes those parts combine while changing their relative positions in particular cases: in this sum of constant elements we find the inner and essential cohesion of the whole, and we expect it to determine the possibility and the manner of variable accretions. The first of these methods, that of bringing out the common element in different instances when at rest, is the one which has been usually followed by logic, and has led to the formation of the logical universal; I should give the preference to the other, that

of determining the element which maintains itself in the same instance under changed conditions; for it is only the assumption that the group *a b c*, the common element in several groups of ideas, will also be found thus to maintain itself, which strictly justifies us in regarding these coexisting elements as coherent, and as the ground for the admissibility or inadmissibility of fresh elements.

23. Abstraction is the name given to the method by which the universal is found, that method being, we are told, to leave out what is different in the particular instances compared and to add together that which they possess in common. If we look at the actual procedure of thought, we do not find this account confirmed. Gold, silver, copper, and lead differ in colour, brilliancy, weight, and density; but their universal, which we call metal, is not found upon comparison by simply leaving out these differences without compensation. Clearly it is no sufficient definition of metal to say negatively, it is neither red nor yellow nor white nor grey; the affirmation, that it has at any rate some colour, is equally indispensable; it has not indeed this or that specific weight, this or that degree of brilliancy, but the idea of it would either cease to have any meaning at all, or would certainly not be the idea of metal, if it contained no thought whatever of weight, brilliancy, and hardness. Assuredly we do not get the universal image of animal by comparison, if we leave out of our minds entirely the facts of reproduction, self-movement, and respiration, on the ground that some animals produce their young alive, others lay eggs, others multiply by division, that some again breathe through lungs, others through gills, others through the skin, and that lastly many move on legs, others fly, while some are incapable of any locomotion. On the contrary, the most essential thing of all, that which makes every animal an animal, is that it has some mode or other of reproduction, of motion, and of respiration. In all these cases, then, the universal is

produced, not by simply leaving out the different marks p^1 and p^2, q^1 and q^2, which occur in the individuals compared, but by substituting for those left out the universal marks P and Q, of which p^1 p^2 and q^1 q^2 are particular kinds. The simple process of leaving out only takes place when one of two individuals compared actually possesses no species of a mark P, of which some species is a necessary mark of the other. Thus we suppose, whether rightly or wrongly does not matter, that we cannot find in plants any trace of sensation and self-movement, both of which are essential to all animals; we do therefore form the universal idea of organic being from a comparison of plant and animal by leaving out these marks without compensation. If we went thoroughly into the facts, we should perhaps find occasion, not indeed in this instance but in many similar ones, to continue to ascribe two marks jointly to both the objects compared, but to assume them to be at zero in the plant, while in the animal they always occur in an appreciable quantity. To express the matter somewhat differently, it may be asserted from the point of view of logic that compensation by the corresponding universal for omission of individual marks is the regular rule of abstraction, while the uncompensated omission applies to exceptional cases, where we can find no logically common mark, of which the presence and absence of some individual mark might be held to constitute different species. So formulated, our rule of abstraction covers these cases of mere omission; on the other hand, a rule which made omission its sole starting-point could find no way to bring in compensation afterwards; and the importance of compensation in forming the universal will be confirmed at every step in the later stages of logic.

24. After the considerations urged in the preceding section, the necessity of which to what was to follow will now be clear, the apparent circle involved in the injunction to form universals by putting together universals, will not

give serious offence. We have seen that the universal marks P and Q which we require here, the 'first universal' of the section referred to, come to us without logical effort as simple facts of observation in our mental life; and just for this reason they can be applied in building up this second universal, which we do produce by logical effort. That the yellow of gold, the red of copper, and the white of silver are only variations of a common element which we proceed to call colour, this is a matter of immediate sensation; but to a person who could not be made sensible of it, it could never be explained by logic either that these particular impressions are species of this universal, or what is meant by a universal as such and the relation of its particular to it. It is just this point to which I would again draw attention here, that the immediate perception of a first universal and the application of some kind of quantitative ideas is the condition of the formation of the second universal in all cases, not only in those like metal where there is no difficulty in regarding the marks of colour, brilliancy, and hardness as stable properties of that which they describe, but also where, as in the case of the animal powers of reproduction and motion, they are merely short adjectival descriptions of conditions which we cannot think completely but by means of manifold relations between various related points. It is easy to convince oneself by an analysis (which I only leave the observant reader to make for himself because it threatens to be a long one) that all differences between animals, even in these respects, issue ultimately in quantitative determinations, whether of the force with which some identical or similar process takes place in them, or of the number of related points between which it takes place, or of the variations in form to which it is liable owing to variations in the number of these related points, the intimacy of their relations, and their relative positions in space and time, these last, like the rest, being measurable variations. If

we take away this quantitative gradation and comparability, which extends, though of course in different ways, to everything, whether simple properties, or their relations, or combinations of events simultaneous or successive, the formation of a universal by comparison of different groups of ideas, would, at least in the sense in which it has any value for thought, be impossible.

25. I will now mention some traditional technical expressions. If we provisionally give the general name of concept (*notio*, *conceptus*) to the composite idea which we think as a connected whole, the sum of individual ideas or marks (*notae*) a, b, c, d, etc., through which a concept S is fully thought and distinguished from all other concepts Σ, is called its 'content' (*materia*); while its 'extent' (*ambitus*, *sphaera*) is the number of individual concepts s^1, s^2, s^3, etc., in each of which the content of S, that is, the group of marks a, b, c, d, in some one or other of their possible modifications, is contained. The colour, a, weight, b, elasticity, d, and the like, would together form the content of metal, S, while copper, s^1, silver, s^2, gold, s^3, and the like, taken together, form its extent. It is usual also to speak of the individual marks a, b, c, as 'coordinated' in the content of S, and of the individual species s^1, s^2, s^3, as 'coordinated' in the extent of S: the relation of the species s^1, s^2, s^3, to the universal itself which forms their genus, is called 'subordination,' while both the species and the genus are said to be 'subsumed' under each of the universally expressed marks, which make up the content of S, and consequently also of s^1, s^2, s^3. Lastly, it is asserted that the extent and content of every concept vary inversely; the greater the content, that is, the number of marks which the concept imposes upon all its subordinate species, the smaller is the number of species which fulfil this requirement; the smaller the content of S, the greater is the quantity of individuals possessing the few marks necessary to make them species of S or bring them within its extent.

If therefore we compare the universal concept S with a similar universal T, and look for a third universal U to which both of them belong as species, and if we continue this process, the higher each universal concept W stands in the scale, that is, the farther it is removed from the concepts S and T originally compared, the poorer will it be in content and the larger in extent; and conversely, if we descend from the highest universals W through V and U, S and T, to the species of S and lower, the content will increase with the decreasing extent and become greatest in those completely individual ideas to which logic hesitates to give the name of concept at all.

26. The value of these distinctions is unequal, but on the whole slight. I will begin what I have to say about them by fixing the terminology which I shall myself use in future. I speak of any composite matter s as conceived or as a concept, when it is accompanied by the thought of a universal S, which contains the condition and ground of the coexistence of all its marks and of the form of their connexion. After this explanation we shall not hesitate to speak of concepts of perfectly individual things (singular concepts, in the old logical terminology), and we believe this to be quite consistent with the usage of language. For when we observe a new object s for the first time, and, not content with the perfectly clear sensible perception of it, go on to ask what it really is, we clearly want to know the rule which connects the perceived marks in the observed fact and converts them into a coherent whole of a definite and predictable character. If we then find that this s is S, an animal or a plant, we suppose ourselves to have a conception of s; it is the idea of it which is raised into a concept by the accompanying thought of the universal S. Every proper name is an illustration of this. 'Alcibiades,' for human thought, never means merely a multiplicity of differently coloured points, which are combined in space in a definite though not quite invariable outline, and resist the

attempt to separate them; nor does the name express merely the accessory notion that this multiplicity in some unexplained way forms a whole; it suggests to the mind a definite general image of a man or a human being, which lays down the lines for our view of the connexion of the observed marks with one another and with the future behaviour to be expected from them. A view so determined cannot be appropriately called either a perception, or an idea merely, but only a *singular concept*.

27. On the other hand, it seems to me quite out of place to call the universal *S* itself, the accompanying thought of which makes the individual into a concept, without any reservation a universal *concept*. *S may* have the form of a concept, but by no means always has it; often it remains a mere general image, the thought of which is indeed accompanied by the thought of its connected wholeness, but does not exhibit the organic rule of the connexion. The name 'man' as ordinarily used expresses no more than an image of this kind; reflexion, by subordinating it to the universal 'animal,' easily makes it into a concept; but then 'animal' remains a general image, which only the naturalist, for the uses of his science, converts into a concept by thinking 'organic being' along with it. It is upon such incomplete logical activity, which brings into relief only a single link in the chain, the connexion of the individual with its nearest universal, but leaves all beyond it in darkness, that the concepts which occur in ordinary thinking are based; as however scientific investigations, to which logic is primarily intended as an introduction, do really aim at extending the conceptual form from the concept itself to the higher universals under which it successively falls, it is enough to have made the above remark without rigidly enforcing it, and I shall follow ordinary usage in conceding the name of concept to those general images as well. I can do this the more easily because the name 'concept' does not seem to deserve in logic that exalted significance which the

school of Hegel has given it, and in which it claims to express the knowledge of the essential nature of the object. The difference between logical forms and metaphysical ideas must be taken into account here as elsewhere. There may be a privileged concept, which follows the thing itself in its being and development, or takes up a point of view at the very centre of the thing, the fountain-head of its self-determination and self-organisation; but it is not the function of logic to reserve its concept-*form* for so very select a filling. By the logical concept we understand such a form of apprehending any matter of thought, from whatever point of view, that consequences admit of being drawn from it which coincide again at certain points with results flowing from that matter, that is, from the thing itself; and as the thing projects itself differently at every different point of view, there may be various equally right and equally fruitful logical concepts of the same object. We may therefore continue to call 'concept' any apprehension which, though only with the help of a general image which is not further analysed, has the effect of bringing the given object under a rule of behaviour which agrees, when applied, with its actual behaviour.

28. The asserted coordination of marks in the content of the concept raises serious difficulties. To begin with, it is a misfortune that we have no appropriate name for the elements of which we compose the concept; for 'mark' and 'part' only apply in certain cases. They give rise to the current delusion that the elements of a concept are universally of equal value, connected in the same way each with the whole and each with each. The ordinary instances of logic, taken from simple natural objects, are specially calculated to lead us into this error. It is true that gold is yellow only in the light, ductile only under a certain power of traction, heavy only for the body upon which it presses; but these various modes of behaviour easily present themselves to our imagination as stable properties, collected in a

definite point of space, and inhering, in a manner identical but otherwise unexplainable, in the reality which on their account we call gold. Here the name 'marks' is appropriate, and here the marks are certainly coordinated in the content as has been asserted; but this coordination merely means that they are all equally indispensable to the whole, but have *not* any other sort of order. If we leave such simple instances, and consider concepts like 'triangle,' 'animal,' or 'motion,' we require, in order to think them properly, a quantity of part-ideas which are no longer mutually equivalent, but have to be placed in the most various relations to one another. The three sides of a triangle are not merely there *as well as* the three angles; they must form the angles by their intersections: the concept of motion does not merely contain the part-ideas of place, change, direction, and speed; direction and speed are, each in a different sense, determinations of change; place, being that which is left behind, can least of all be called a mark of the concept; it is a point of reference for the idea of change, to which its relation is expressed by that of the genitive to the nominative which governs it. To follow out these points in detail would take too long, but it would evidently lead us to the conviction that, as a rule, the marks of a concept are not coordinated as all of equal value, but that they stand to each other in the most various relative positions, offer to each other different points of attachment, and so mutually *determine* each other; and that an appropriate symbol for the structure of a concept is not the equation $S=a+b+c+d$, etc., but such an expression as $S=F(a, b, c, \text{etc.})$, indicating merely that, in order to give the value of S, a, b, c, etc., must be combined in a manner precisely definable in each particular case, but extremely variable when taken generally. If in any particular instance $S=a[b^{c} \sin d]+(e-\frac{f}{g})\sqrt{h}$, this formula, however foolish it would be if it professed to mean anything

more, would give a better picture than the above inadequate formula of addition of the different ways in which the several marks *a*, *b*, *c*, etc. contribute to the construction of *S* as a whole.

29. No objection need be made to the coordination of s^1, s^2, s^3, copper, gold, silver, within the sphere of *S*, metal; on the other hand, attention should be drawn to the great difference of value between the subordination of the species to the genus, and that of the universal *S* along with its species to the universal marks *a*, *b*, (ductile, coloured, etc.). The nature of the universal, metal, completely dominates the nature of its species, gold and copper, and no property of the latter escapes its influence; many things are yellow or red, but the glistening red and yellow of copper and gold belong to metal alone; many things are ductile, but the amount and other peculiarities of the ductility exhibited by gold and copper are heard of only in metals; and only metallity explains their degree of specific gravity. Similarly the universal animal determines every property and every movement of its species; animals move, grow, and rest differently from plants and lifeless things. If we symbolise the universal metal by a circle *S*, the smaller circle of gold, s^1, lies entirely within it, and by the side of this, separate from it but also completely inside *S*, the circles s^2, s^3, copper and silver. Applying differently two names which are generally used as equivalents, I describe the true subordination to a dominant universal as *subordination* to the genus, while I call the subordination of gold to yellow or ductile *subsumption* under the mark. These universal marks obviously do not rule and penetrate the whole nature of gold; each of them expresses only one side of it, which it shares with other objects of an entirely different kind, from which, so far as logic can see, no sort of inferences can be drawn as to the other properties of gold. Thus the lesser circle *s*, gold, occurs only in a particular place in the larger *G*, yellow, and intersects it without lying wholly within it; *G*

is similarly intersected in other places by the circles of other yellow objects, and they all remain partially outside it.

30. Starting from the universal S, which was the rule for s^1, s^2, s^3, the original objects of comparison, we were able to mount to higher and higher universals T, U, V, W. In natural history, where such a series is of value, its several members in an ascending scale have been named species, genus, family, order, class: there is however a difference of opinion as to what functions a universal concept must perform in order to represent even a species or a genus, and the other names are applied still more divergently, and always from points of view depending for their justification on the special nature of the subject-matter. If we dispense with this plea, the plea from the side of the specialist for the significance and importance of these distinctions, the only way to give some sort of fixed logical value to species and genus is as follows. The only thing which suggests to the natural mind to look for a universal, is the comparison of individual instances which are not identical but similar. To seek for a concept which included under it cucumbers and mathematical principles, could only be an ingenious joke; but all varieties of human beings, big and little, old and young, fat and thin, black and white, provoke the natural mind to the search. Their sensible appearances produce similar images, at the corresponding points of which only such marks occur as are immediately felt to be species of the same universal mark, such as hardness or colour; and the relations between any two of these points are in all cases merely modifications, differing in degree and amount, of one and the same universal relation. The comparison of individual men, therefore, produces a universal image; not indeed in the sense that the universal man can really be painted, but in the sense of the illustrations in a natural history, which purport by one camel or horse to exhibit all camels or horses clearly to perception, in a form which is more than a mere scheme or symbol; or again in the sense

of geometry, in which a drawn triangle, though necessarily individual with others existing beside it, yet represents all these others, and in a similarly perceptible form. But this possibility vanishes when we ascend to higher universals, in which these universal images are themselves included in their turn as species: the universal mammal, which is neither horse nor camel nor is otherwise named, cannot even be drawn in a schematic form, any more than the polygon can which has neither three, four, or any other definite number of sides. Thus these higher universals are no longer apprehended in perception, but only in thought, by means of a formula or equation, which prescribes essentially the same relation between various related points, but leads to quite different perceptible configurations, accordingly as the previously undetermined values of these points and their various connexions are differently determined in thought. I would then call a universal which still admits of an image, a species, and the first of those which can only be expressed by a formula, a genus, in agreement, as I believe, with the instinct of language, and incidentally also with the old terminology of Aristotle; for in his choice of the words *εἶδος* and *γένος* he was no doubt determined by their original meanings; *εἶδος*, the species, which includes only individuals under it, is the common element in the look or appearance of things, while *γένος* comprehends things which differ in form, but in their process of growth, or, if they have no growth in time, in the regulative connexion of their parts, obey the same law and formula.

31. It remains to consider the last of the assertions mentioned above, that of the inverse ratio between the content and extent of concepts; this seems to me to be untrue where its truth would be important, and to be comparatively unimportant where it is true. The number of marks, of which we compose our concepts, is not infinite; the words of language, numerous but not innumerable, suffice to de-

note them. It may therefore easily happen that a group of them, say *i k l.* occurs in several universal concepts, *S T* and *V*, at once, without its therefore representing a higher universal containing all species of *S T* and *V*. We may class cherries and flesh under the group *i k l* of red, juicy, edible bodies, but we shall not suppose ourselves thereby to have arrived at a generic concept of which they deserve to be called species. I do not say that in giving exclusive prominence to such groups there is always as little sense as in this absurd instance; we shall see later how valuable the process may be; it helps to show, what is often useful and necessary, that different subjects, though otherwise quite foreign to one another and not subsumable under any common generic concept, are nevertheless, in consequence of a single or a few common marks, jointly liable to certain inevitable consequences. If then anyone chooses to go on to call these groups of marks universal concepts, he is certainly right about the inverse ratio of their content and extent: the fewer members there are in the group, the more sure will it be to occur in all sorts of concepts, and again, the greater the number of different ideas compared, the smaller will be the group of marks in which they all agree. Of the true universal, on the other hand, which contains the rule for the entire formation of its species, it may rather be said that its content is always precisely as rich, the sum of its marks precisely as great, as that of its species themselves; only that the universal concept, the genus, contains a number of marks in a merely indefinite and even universal form; these are represented in the species by definite values or particular characterisations, and finally in the singular concept all indefiniteness vanishes, and each universal mark of the genus is replaced by one fully determined in quantity, individuality, and relation to others. It is true that instances may be alleged against the universal validity of this assertion, like that mentioned above of organic being, to the concept of which we subordinate plants and animals;

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it may be called a logical caprice to retain the marks of sensibility and motivity in this concept, with the tacit reservation that they are both at zero in plants. But what this instance properly shows is rather, that the higher universals, from the genus upwards, really cease to be true universal *concepts*, and pass over into groups of conditions, imposing uniform consequences upon various genera, more properly so called. The concept of organic being is such a group of marks, $i\ k\ l$, which does not occur in any independent form of its own, but in the genera in which it does occur, plants and animals, gives rise necessarily to the same results.

32. By the preceding remarks I neither hope nor aspire to bring about a permanent change in the traditional terminology: they were intended merely as helps to a clearer insight into the structure of concepts in general. With the same object I add the following. I express the genus G, so far as its concept gives the rule of combination for a number of individual marks $A\ B\ C$, etc., by $F\,[A\ B\ C]$, and I assume that each of the marks admits of particular forms, which we may call $a^1\,a^2\,a^3\,.\,.\,b^1\,b^2\,b^3\,.\,.\,c^1\,c^2\,c^3$; also that the principle of combination F has freedom to assume various forms, of which we may indicate three by f, ϕ, and $\mathfrak{f}$. Now as the marks $A\ B\ C$ may be of very different value for the whole G, it is possible that the different values assumed e.g. by A may be of decisive importance for the configuration of the whole, and may also exercise a transforming influence upon the combination of the other marks. The consequence of this may be that, as A assumes one or other of its values, the organisation of the whole, F, changes from one of its particular modes to another; the sum total of the species of G would then be,

$$G = f\,(a^1 B\ C\ldots) + \phi\,(a^2 B\ C\ldots) + \mathfrak{f}\,(a^3 B\ C),$$

omitting for shortness' sake to express the corresponding changes in B and C. These decisive marks, $a^1\,a^2\,a^3$, are in this case the *specific* differences, *differentiae specificae*. Thus Aristotle, who gives them the name of διαφορά, when he sub-

ordinates man to the genus animal, usually describes the faculty of rational thought as that peculiar characteristic, a^1, of the universal psychical life, A, by which man is distinguished from all other animals: to this we may now add, following out what I have indicated above, that this a^1 not only separates man from brutes, but also determines the values of B and C peculiar to him, as also the mode of their combination, i.e. the general character by which man is distinguished from the brutes with their peculiar organisation ϕ or f. It may further happen that the particular values which one or more of the generic marks have assumed in a single species, are possible in this and no other species, and that yet they have no important influence upon the shaping of its other marks, and do not therefore represent the nature of it in all its aspects. Such a mark is called by Aristotle property, ἴδιον; it is what we call a characteristic; Aristotle gives risibility as a property of man, Hegel, in a similar sense, the ear-lap; both distinguished man from the brutes, but without exhausting his nature. There are also, according to Aristotle, marks which do not belong to the rigid constitution of a concept, but indicate something which comes in contact with or happens to it; every verb which says, e.g. Socrates 'is sitting' or 'standing,' is an example. Translators torment themselves in vain to find an equivalent for both the real and the etymological sense of Aristotle's expression συμβεβηκός; what is important and true in it answers completely to what we call *state*; that this word does not nevertheless cover the usage of Aristotle seems to me to be the fault of an inexactitude of his own, which it is scarcely worth while to enter into. As to the relation in fact between the concept as a whole and this species of mark, its consideration belongs to the theory of the judgment. In the introduction of Porphyrius to the Aristotelian logic there is material enough for further reflexion, though indeed of a mostly unprofitable sort, about the likenesses and differences of the logical determi-

nations here touched upon; we have used them primarily to illustrate the complex organisation of concepts, and with this view have not always agreed with Aristotle in the form of our exposition.

33. And now, where do we get to at last if we go on looking for higher and higher concepts above those which we have already found? What form does the entire system of our concepts assume if we suppose this task completed? It must be a structure resting on a broad base, formed by all singular concepts or ideas, and growing gradually narrower as it rises. The ordinary view, in fact, gives it the form of a pyramid, ending in a single apex, the all-embracing concept of the thinkable. I cannot see much point in this notion; it rests entirely upon that unmeaning subsumption under a mark, the logical value of which we have already depreciated. A single step suffices to bring everything at once under the head of the thinkable; we may spare ourselves the trouble of climbing up to this result by a pyramidal ladder; and moreover the result itself ignores in the most absolute and unmeaning way everything which gives substance and character to thought. If on the other hand we follow the method of subordination to the genus, and arrange the manifold only under such universals as still imply the notion of universally regulating its specific conformations, we arrive not at one but at several ultimate concepts not reducible to one another, in which we are not surprised to recognise those very meanings of the parts of speech which at the outset we found to be the primary logical elements. All substantives go back to the radical concept of something, all adjectives to that of quality, verbs to that of becoming, and the rest to that of relation. It is true that all these radical concepts have the common mark of being thinkable; but there is no common genus over them of which their several essences form species, nor does any one of them occupy this position in regard to the rest; it is not possible to apprehend some-

thing as a species of becoming, or becoming as a species of something. From this point of view the entire structure of our concepts rises like a mountain-chain, beginning in a broad base and ending in several sharply defined peaks.

Transition to the form of the Judgment.

34. It was this image of a conceptual world building itself up without a break, upon which the vision of Plato dwelt. The first to recognise the eternal self-identity of every concept and its significance as against the variableness of the real world, he might well feel the charm of tracing out all the simple elements of thought, of combining all that could be combined, and of setting up in the organic whole of a world of ideas the eternal pattern of which the created world is an imperfect imitation. But neither he nor his successors have attempted actually to execute this essentially impossible task: still less should we now be inclined to regard its execution as desirable. And this not only because reality, things as they are, suggests riddles too many and too hard to leave us any time for drawing up an inventory of what might be but is not; for even a perfect knowledge of the ideal world would give us little support in understanding the real. The utmost that we could attain by such means would be merely the image of a fixed order, in which simple and composite concepts stood side by side, each unchangeably self-identical and each bound to its place in the system by invariable relations to all the rest; whereas what reality shows us is a changing medley of the most manifold relations and connexions between the matter of ideas, taking first one form and then another without regard to their place in the system. This great fact of change does not cease to be a fact because, in the spirit of antiquity, we find fault with it as an imperfection compared with the solemn rest of the world of ideas: the

current of our thoughts is perpetually bringing it before us again, and the mind, receiving as it does from that current the stimulus to activity, has to exert itself to reduce even these changeable coincidences to principles of coherence. The next advance of logic is determined by this fact.

35. There are different considerations which lead us to take the same step next. When new marks, of which we were not before conscious in a concept, attach themselves to it without its apparently being changed, we are directly stimulated to ask what ground can be conceived for such a variable connexion of the two. But also when we compare different instances of a universal, in the universal marks of which we have already included the possibility of many particular ones, it may still be asked on what ground a particular mark in each instance coheres with the rest of the content, and why this particular mark is privileged above all the others which remain absent, though, as species of the same universal, they might equally well be present. Lastly, as we think of every concept as uniting a number of marks, and these marks, though not essentially related as members of one and the same systematic series, but rather heterogeneous and foreign to one another, nevertheless determine each other and in their combination influence the accession of others, the question again recurs, what is the ground of the apparent coherence in this coexistence of heterogeneous elements. We are conscious that when, in considering the concept, we attributed to a certain combination of marks this position of a dominant logical substance, operating in a number of different or changing forms, we required and presupposed a view which we have yet to show to be logically practicable. This then is our present problem, either to break up these presupposed combinations again, or, if they can be justified, to reconstitute them, but in a form which at the same time expresses the ground of coherence in the matter combined. In seeking to solve this problem, the form in which thought

will move will obviously be that of the *judgment*. In this a permanent conditioning member, the whole content of a concept, appears as *subject* over against the variable or conditioned members or the sum of them, as *predicates*; the relation of the two, explaining and justifying their connexion, lies in the *copula*, that is, in the accessory notion which, more or less fully expressed in language, holds together the two members of the sentence.

CHAPTER II.

The Theory of the Judgment.

Preliminary observations on the meaning and customary division of Judgments.

IN accordance with the general plan of my exposition, I should now have to develop the various forms of judgment systematically as members of a series of intellectual operations, each one of which leaves a part of its problem unmastered and thereby gives rise to the next. Before beginning this attempt, I must say a few words about other usual modes of treatment, and my reasons for deviating from them.

36. Every judgment formed in the natural exercise of thought is intended to express a relation between the matters of two ideas, not a relation of the two ideas themselves. Of course some sort of relation between the ideas follows inevitably from the objective relation in the matter which they represent; but it is not this indispensable relation in the mental media through which we endeavour to grasp the matter of fact, but this matter of fact itself, which is the essential meaning of the act of judgment. When we say, 'gold is yellow,' it is indisputable that in this judgment our idea of gold lies within the sphere of the idea of yellow, and that accordingly the predicate is of wider extent than the subject; but it was certainly not this that we intended to express by the judgment. We wanted to say that yellow itself belongs as a property to gold itself, and only because this relation of fact is already presupposed

to exist (whatever difficulties this may involve), can it be reproduced in a sentence in which the idea of gold is contained by that of yellow. Logic indeed has already drawn attention to the fact that we are not quite right even in making this sentence; appealing from what we express to what we mean, it teaches that the subject also from its side limits the too extensive predicate; gold is not yellow simply, but golden yellow, the rose rosy red, and this particular rose only this particular rosy red. But even with this correction the imperfection of this whole view of the judgment is not mended; for it does not tell us what is after all the relation between the two members so corrected, and it loses sight entirely of the great possible variety in the modes of their connexion. Thus gold is not yellow in the dark; its colour therefore only attaches to it under a condition, that of the presence of light; and if we wished to connect this new experience with the previous one in the phraseology of the view which we are now considering, we should have to say, the idea of gold lies simultaneously within the spheres of that which is yellow in the light and of that which is not yellow in the dark; but this form of expression seems to me only to betray a disposition to leave the really important point, the mention of the conditional relation, and to go off upon results which are true but quite without significance. Doubtless these relations of extension between the ideas combined in the judgment have their logical value; but where the want of them is felt, they are not so difficult but that they can be mastered at the moment without special effort: to give them a chief place in the consideration of the judgment seems to me to be as erroneous as it is wearisome.

37. The technical expressions of logic point to the view which I have taken here. In the judgment above the *subject* in the sentence, that is, the grammatical subject, is the *word* gold, the subject in the judgment, the logical subject, is, not the idea of gold, but gold; for it is to this only that

yellow belongs as that which is predicated of it, and predicated in a definite sense indicated by the copula. On the other hand, the idea of yellow is not a property of the idea of gold in the same sense in which yellow is of gold; the one idea is not affirmed or predicated of the other; the relation which exists between them is primarily no more than this, that whenever, or whenever under certain conditions, the one idea, gold, is found, there the other idea, yellow, is also found, but that the former is not always present when the latter is. But to explain and express what it is which makes this relation possible, justifiable, or necessary, is the problem of the logical judgment alone, and it solves the problem by exhibiting through its copula the relation between the object-matters of the two ideas, a relation due to that which the ideas represent and differing in different cases. On the other hand, it is only between these object-matters that a logical copula is conceivable; between the ideas there is no relation but that of the psychological connexion mentioned above, and that of the monotonous, unmeaning inclusion of the one within the other.

38. It is now clear that for us there can be only so many essentially different forms of judgment as there are essentially different meanings of the *copula*, that is, different accessory notions which we form of the connexion of the subject with its predicate, and to which we give more or less complete expression in the syntactical form of the sentence. Thus many other distinctions which meet us in logic have no use or place in our systematic survey, though they may still have a logical value of some other kind. To secure clearness in what is to follow, therefore, it is desirable to give a preliminary explanation of traditional views; but I think I may confine it to that division of judgments to which Kant has given currency in Germany, though it is itself of much older date. According to Kant, as we know, the character of every judgment is determined in four respects, quantity,

quality, relation, and modality, and in each respect every judgment has necessarily one of three mutually exclusive forms. I may exclude the third member of this division from these preliminary considerations, for *relation* (between subject and predicate), in respect of which Kant distinguishes categorical, hypothetical, and disjunctive judgments, clearly concerns just those essential characteristics of the judgment which we are looking for, and which I shall have subsequently to expound myself. If the categorical judgment connects its subject *S* and its predicate *P absolutely*, as the phrase is, or on the simple model of the relation of a thing to its property, while the hypothetical assigns *P* to *S*, not immediately, but only on the assumption that a certain condition is fulfilled, and the disjunctive gives *S* no definite predicate, but imposes on it the necessity of choosing between several mutually exclusive ones, there is no doubt that in each of these three forms the sense of the copula, the mode of connexion between *S* and *P*, is different and peculiar; these three will form the series of judgments which we shall have subsequently to construct; only the nine remaining ones call for the following preliminary remarks.

39. In respect of their *quantity* judgments must be either *universal* or *particular* or *singular*. If we express these distinctions by the usual formulæ, 'all *S* are *P*,' 'some *S* are *P*,' 'this *S* is *P*,' it is clear that they indicate merely the different extents to which a connexion between *S* and *P* is supposed to hold good; the nature of the connexion in all the cases is the same, and must be the same, because the universal judgment, according to this view of its meaning, admits of being formed by summing the singular and particular ones, and must therefore be perfectly homogeneous with them. Thus the quantitative description applies to the subject only, and has no reference to the logical relation between it and its predicate; it is therefore of importance where the connexion of ideas requires the

application of a judgment, the import of which depends upon the circuit over which it holds good; but no special advance in logical activity is indicated by these distinctions as they are here formulated. I say 'as they are here formulated,' because certainly the quantitative differences of judgments are really connected with important logical differences in the mode of connexion between *S* and *P*; for doubtless that which belongs to all *S* has also a different hold upon the nature of its subject from that which belongs only to some; but the quantitative formulation of the judgment, which merely *counts* the subjects, just fails to seize this important accessory notion, and makes the relation of the predicate to its subject, often in violation of the fact, appear the same in all cases.

40. In respect of *quality* Kant distinguished *affirmative*, *negative*, and *limitative* judgments. Nothing is clearer than that the two sentences '*S* is *P*,' '*S* is not *P*,' so long as they are supposed to be logically opposed to one another, must express precisely the same connexion between *S* and *P*, only that the truth of that connexion is affirmed by the one and denied by the other. It is useful, though certainly not necessary, to make this clear to ourselves by splitting each of these judgments into two. We think of a certain relation, whatever it may be, between *S* and *P* expressed in the judgment '*S* is *P*' as an idea still open to question; this relation forms the object-matter upon which two opposite judgments are passed; the affirmative gives it the predicate of validity or reality, the negative refuses it. In the connexion of our thoughts it is of course of the greatest importance which of these judgments is subsequently passed upon a given connexion between *S* and *P*; but this difference does not give rise to two essentially different kinds of judgment as such; validity or invalidity are rather to be considered, in regard to the question before us, as predicates of fact to which the whole content of the judgment forms the subject. This content itself can be expressed in a form as yet neither

affirmative nor negative in the interrogative sentence, and this indeed would take the third place amongst the three qualities of judgment more appropriately than the limitative or infinite judgment, which is supposed to attribute a negative predicate to the subject by a positive copula, and is usually expressed in the formula '*S* is not-*P*.' Much acumen has been expended even in recent times in vindicating this form of judgment, but I can only see in it an unmeaning product of pedantic ingenuity. Aristotle himself saw clearly enough that such expressions as 'not-man' are no concepts; they are not even apprehensible ideas. The truth is that, if 'not-man' means all that it ought logically to mean, that is, everything that is not man, triangle, melancholy, sulphuric acid, as well as brute and angel, it is an utterly impossible feat to hold together this chaotic mass of the most different things in any *one* idea, such as could be applied as a predicate to a subject. Every attempt to affirm this unthinkable not-*P* of *S* will be found by an unsophisticated mind to end in denying the thinkable *P* of the same *S*; instead of saying, 'spirit is not-matter,' we all say, 'spirit is not matter.' Even in cases where in natural thinking we seem really to make a limitative judgment, as e.g. when we say 'doctors are non-combatants,' we are in truth making only a negative one. For this not-*P* has not here the meaning which the limitative sentence would give it; according to that, horses, wagons, triangles, and letters would be non-combatants; what is meant is only human beings who belong to the army but are declared to take no part in fighting. Thus there is never any necessity to the natural mind for forming limitative judgments; every inference which could be drawn from '*S* is not-*P*' can also be drawn from '*S* is not *P*.' It is not worth while to spend more words on this point; obvious vagaries in science must not be propagated even by a too elaborate polemic.

41. Through the forms of *modality* different values are supposed to be given to the relation which is conceived

to hold between S and P; the *problematic* judgment expresses it as merely possible, the *assertorial* as real, the *apodeictic* as necessary. But these new properties are treated quite independently of the way in which judgments have been already determined from the other three points of view. After it has been fixed whether a given judgment connects its elements in categorical, hypothetical, or disjunctive form, after it has been decided whether it affirms or denies the relation conceived in one of those forms, and after the extent of the subject to which the predicate applies has been limited by the expression of quantity, it is still held to be an open question whether the judgment so composed will be problematic, assertorial, or apodeictic. To treat the matter thus is to confess openly that the possibility, reality, or necessity, spoken of here, stand in no connexion with the logical construction of the judgment. All these judgments, which are usually expressed in the formulæ 'S may be P,' 'S is P,' 'S must be P,' are entirely the same as regards the validity which they give to their contents by logical means; they are all merely assertions of the person who makes them, and are distinguished only by their object-matter. This, the possibility, reality, or necessity of a relation between S and P, they express either without any grounds at all, or upon grounds derived from right reflexion upon the facts, which they do not then allow to appear in any way in their logical structure; just for this reason they need additional auxiliary verbs, in order to express independently what does not lie in the form of the judgment itself. In more developed connexions of thought such judgments of course have their value; for what is wanted is often to compress results of previous reflexion into the shape of simple assertions, without perpetually repeating the grounds upon which they rest; here these auxiliary verbs are in place, expressing in the form of a now familiar fact the possibility, reality, and necessity which once had a logical justification. But for the separation of

essential forms of judgment and their systematic arrangement, the only modality that could be of value would be one which, instead of going its own way independently of the logical nexus of the other judgments, grew out of that nexus itself, and expressed the claim to possible, real, or necessary validity, which the content of the judgment derives from the mode in which its elements are combined.

42. It would be useless to ask for such a modality, if we could not show the possibility of it. I will therefore anticipate somewhat what I have to say later. The proposition, 'all men must die,' is usually held to be apodeictic; I consider it merely assertorial; for it states only, and does not give grounds for, the necessity of which it speaks; so far as its form goes it does not even decide whether all men die for the same reason, or everyone for a special reason, so that the various conditions agree merely in the fact that they leave no one alive. And yet what we had meant by the sentence was, not only that all men as a matter of fact die, but that the extension of mortality to all has its ground in the universal concept of man, in the nature of humanity; and this thought we do in fact express by the general form of the judgment 'man dies'; for the sense of this judgment, the difference of which from the ordinary universal I shall come back to, is not of course that the universal concept man dies, but that everything dies which is included under it, and for the reason that it is so included. Every hypothetical judgment, again, gives in its protasis the ground for what is stated in its apodosis, and is therefore in my sense an apodeictic form of judgment; the apodosis here is not simply asserted, but asserted conditionally upon the validity of the protasis; but, presupposing that validity, the content of the apodosis is no longer a mere fact, but a necessity, with the same right with which every consequence necessarily follows from its conditions. Similar remarks might be made, if they would not be too long for this preliminary section, about the

disjunctive judgment; and thus we should have found in the three forms of relation three forms also of apodeictic modality.

43. I will guard myself against a misunderstanding, though it would be so gross that I am almost ashamed to do so. The form which we give to the content of a judgment can never guarantee its truth to fact; this always depends upon whether the relations between the elements of the content itself are truly such as the form of the judgment, in order to ascribe to them a certain sort of validity, has to presuppose. This holds good of the ordinary modality no less than of that which we would put in its place. In the ordinary form of the apodeictic judgment, '*S* must be *P*,' any nonsense may be expressed without thereby becoming sense; and it is equally open to us to misuse the judgments which I call formally apodeictic, and say 'man is omnipotent,' 'if it rains everything is dry,' 'every triangle is either curved or sweet or hasty-tempered.' These latter forms of judgment, then, do not, any more than the former, make every connexion of concepts which is put into them true or necessary; the significance of them lies merely in showing the formal conditions under which we may ascribe demonstrative certainty to a given content, *if* that content is in itself such as to satisfy them. And here our view of modality differs to its advantage from the ordinary one. The latter merely tells us that there is demonstrative knowledge, and that, if we have got it, we can express it in the form '*S* must be *P*'; but it does not tell us how knowledge must look, and what its internal structure must be, in order to be demonstrative and to justify this expression. Our plan on the other hand does show us this; we find that there are three forms of relation between *S* and *P*, which, when they exist, lead to necessary knowledge; endeavour to bring your ideas into one of these forms; either frame general judgments and look for the *P* which is already implied in the conception of a genus *S*; this *P* then

belongs necessarily to every species of S: or form hypothetical judgments, and show that the addition to S of a condition X gives rise to a P which would not otherwise be present; then this P holds necessarily of every S which comes under the same operation of the same conditions: or lastly form disjunctive judgments; as soon as you have brought a question to a definite 'either . . or,' the thing is settled, and all that is now wanted is experience to determine, in each particular instance, which of two predicates, P or Q, will be true and necessarily true. There are no other ways of arriving at necessary knowledge, and every judgment which we express in the form, 'S must be P,' remains merely an assertion, the matter of which, if it is convincing, has always been originally apprehended in one of those three ways.

44. Thus far I have spoken only of apodeictic judgments: the ambiguity of the ordinary theory of modality is still more striking in the case of *problematic* judgments. The proposition, 'all bodies can be set in motion by adequate forces,' may have any one of the three modalities ascribed to it with about equal right. Firstly, as a statement which does not add the grounds upon which it is made, it is assertorial: but what it states is not a real occurrence, but the possibility of an unreal or only conceived one, and this is enough according to traditional usage to give it the name of problematic: lastly, it may be called apodeictic, because it ascribes a property to all bodies, a property therefore which can be wanting in none and is accordingly necessary to each: in fact, this judgment contains the reality of the necessity of a possibility. From which point of view are we to choose its name? I should be in favour of regarding it as an assertorial judgment, reckoning the necessary possibility as part of the matter asserted. As however the same view may be extended to all problematic judgments of the ordinary form, the question arises whether there is any form of judgment at all

which, as such, deserves to be called problematic. Interrogations and prayers have been alleged as instances, for neither of them really asserts anything; the connexion of *S* and *P* which forms their content seems to be presented to the mind as no more than a floating possibility. I doubt however whether they can be considered as specific logical forms at all. For ultimately interrogation must be distinguished from prayer, and the distinction can only lie in the fact that the conscious attitude of the questioner to his question is different from that of the petitioner to his petition. Suppose the import of the question to be, 'I do not know whether *S* is *P*,' and that of the petition, 'I wish that *S* were *P*'; it would of course be very pedantic to say that the speaker himself must always analyse what he says into this bipartite form, but still, if we take his consciousness as a whole, it must contain in both cases two different states, tempers, dispositions, or whatever we call them, which, *if* we wished to express them, could only be expressed in those ways. If this is so, it is clear that both judgments contain a principal sentence of an assertorial form, which says nothing about the content of the judgment but merely indicates the attitude of the speaker to it; the other and dependent sentence, introduced by the conjunctions 'whether' or 'that,' comprises the whole content, without saying anything about the nature and degree of its validity. It is for this reason that I do not consider the dependent sentence either to be a problematic judgment; for it is not enough that the account of the nature of the import should be merely *absent*; the import ought to be explicitly confined to mere possibility. As to the prayer, it might further be said that it contains the possibility of what is prayed for and nothing else, whereas the question, as it may be a question about possibility itself, does not always do even that: in both moreover the assumption of the possibility of a conceived connexion between *S* and *P* could only be reckoned as a state of the speaker's mind, and would not

lie in the logical form of the judgment. I should rather consider this dependent sentence to express without any modality the mere content of a judgment; and it is just because no complete judgment can be expressed without claiming possibility, reality, or necessity for its import, that these sentences void of modality never occur independently, but are always governed by some other independent sentence which asserts one of those modalities of its content.

45. According to our view those judgments only could be called problematic which by their logical form characterise a conceived relation between *S* and *P* as possible and only as possible. This is done by all quantitatively particular and singular judgments. All that is directly expressed by sentences of the form, 'Some *S* are *P*,' 'Some *S* may or must be *P*,' 'This *S* is *P*' or 'may or must be *P*' is the actual, possible, or necessary occurrence of *P* in certain cases of *S*; they leave it doubtful how the matter stands with the other cases of *S* which are not mentioned; for *S* as such, therefore, it is only the possibility of each of these three relations to *P* which is expressed, and these particular sentences are equivalent to the assertions, '*S* may be *P* possibly,' '*S* may be *P*,' '*S* may be *P* necessarily.' I therefore call particular sentences problematic in respect of the universal *S*; the fact that they are clearly also assertorial in respect of the some *S* of which each speaks, does not at all militate against my view; it only shows us that in fact the only way of recognising a certain relation between *S* and *P* to be merely possible is by observing that the relation does, may, or must hold good of some *S* and not of others. There are therefore certainly no independent problematic judgments, which are not assertorial in respect of a part of their universal subject in so far as they affirm of it a possible, actual, or necessary predicate.

46. Lastly, it is easy to see that, on the one hand, the 'may' and 'must' of the ordinary problematic and apodeictic

judgments and the 'is' of the assertorial by no means suffice to express all material differences of importance in the truth of their several contents, and that on the other hand, just for this reason, they lump together very different relations under the same expression. Firstly, what modality have such sentences as these, '*S* will be *P*,' '*S* ought to be *P*,' '*S* may be *P*,' '*S* has been *P*'? No one of them affirms reality, but the unreal which is past in the last is something quite different from that which is permitted, enjoined, or future, in the others: in the third it is possible, in the second its possibility is doubtful, in the first its reality is inevitable, while in the last it is at once irretrievable and unreal. If all these shades of meaning had been taken into account, the forms of modality might have been correspondingly increased in number. On the other hand, how entirely different in meaning are the similarly formed sentences, 'It can rain to-day[1],' 'The parrot can talk,' 'Every quadrangle can be divided into two triangles.' In the first case we have a supposition which is possible because we know no reason to the contrary; next a capacity which exists upon conditions which need not have existed; lastly a necessary result of an operation which we may carry out or not as we please. I will not multiply these instances, as might be done indefinitely; to attempt to analyse them all would be as foolish as to undertake to work out beforehand all possible examples in a mathematical text-book. In practice, indeed, it is just from these material varieties of meaning in the expressions in question that our inferences are drawn; but we have no resource except to observe in each particular instance what we have before us; whether it is a possibility which may be tentatively assumed in the absence of proof to the contrary, or a well-grounded capacity resting securely upon its conditions; whether it is

[1] ['Es kann heute regnen; der Papagei kann reden'; in English we say, 'It *may* rain to-day,' so that the difference of meaning is represented by some difference of form.]

a necessity due to the presence of imperative reasons, or one arising from a command, a purpose, a duty, or lastly one of those combinations of possibility, reality, and necessity which we touched upon above.

The series of the forms of Judgment.

A. *The Impersonal Judgment. The Categorical Judgment. The Principle of Identity.*

47. There can be no doubt that in the series of the forms of judgment the categorical comes before the hypothetical and the disjunctive. We could have no occasion for making the occurrence of a predicate P in a subject S dependent on a previously fulfilled condition, unless we had already had experiences of the presence of P in some S and its absence in others. Equally little can we think of prescribing to S the necessary choice between different predicates, until previous experiences have established the constant relation of S to a more universal predicate, of which the proposed alternatives are specific forms; and these experiences too would find their natural expression in a judgment of the form 'S is P.' The structure, moreover, of the hypothetical and disjunctive judgments exhibits permanent traces of this dependence: however complex they may be in particular cases, the general scheme to which they are reducible is always that of two judgments of the form 'S is P,' combined, either as protasis and apodosis or as mutually conclusive members, so as to form a single complete assertion. But the question may be raised whether a still simpler form must not precede the categorical judgment itself in the systematic series. The sentence 'S is P' cannot be uttered until the current of ideas has informed us of an S with a fixed position and recognisable character of its own, to which a P can be added in thought as a predicate. Now this will not always be the case;

indeed it may be questioned whether the discovery of the definite *S*, which is to serve as subject to a categorical judgment, does not always presuppose experiences of *S* in a less developed form, and their translation into logical equivalents. This question, which relates to the psychological growth of thought, I leave unanswered here; for our present purpose the fact is enough that even our fully developed thought has preserved a form of judgment which performs this simplest of functions, that of giving logical setting to a matter of perception without regarding it as a modification or determination of an already fixed subject. This is the *impersonal* judgment, which, as the first act of judging, I here treat as a preliminary stage to the categorical.

48. I do not think it necessary to defend at length the logical import of the impersonal judgment against the opinion which would make it merely the linguistic expression of perception itself, without involving any logical activity. The natural sound which a man who is shivering with cold makes when he cowers against another, is a mere sign of this sort, which only serves to give tongue to his feeling; but as soon as he expresses his discomfort in the sentence 'it is cold,' he has undoubtedly performed an act of thought. By giving to the content of his perception, which in itself is undivided, this bipartite form of a predicate related to a subject by a copula, he expresses that he can think of it as a perceived reality in no other form than this. It is true that he is not in a position to give the subject an independent content; he only indicates its empty place and the fact that it requires filling, either by the indefinite pronoun, or in other languages by the third person of the verb, which he uses instead of the infinitive: it is true also that the whole content of the perception which he expresses falls into the predicate alone: and it is true, lastly, that the copula which he puts between them has not as yet the sense of a definitely expressible relation;

it only keeps formally apart what is substantially inseparable and interfused. But it is just by this attempt to bring about an articulation to which the matter of perception will not yet lend itself, that the impersonal judgment expresses all the more clearly the instinct of thought, that everything which is to be matter of perception must be conceived as a predicate of a known or unknown subject.

49. I will now explain why I have here spoken repeatedly of perception[1]. The indefiniteness of the subject in the impersonal judgment has been interpreted to mean that it merely expresses in substantival form what is expressed in verbal form by the predicate. I do not doubt that anyone who is asked what he means by 'it,' when he says 'it rains,' or 'it thunders,' can easily be driven to say, 'the rain rains,' or 'the thunder thunders.' But I believe that in that case his embarrassment makes him say something different from what he really intended by his impersonal judgment. It seems to me to lie in the essence of such a judgment that he really looks upon the determinate matter in question as attaching to an indeterminate subject, the extent of which is much wider than that of the predicate; and if he uses several such expressions one after another, 'it lightens,' 'it rains,' 'it is cold,' though he does not expressly intend to say that the indefinite pronoun means the same in all those cases, he would certainly, if he understood himself correctly, give this answer rather than the former one. This 'it' is in fact thought of as the common subject, to which the various phenomena attach as predicates or from which they proceed; it indicates the all-embracing thought of reality, which takes now one shape, now another. This has been rightly felt by those who found in the impersonal judgment a judgment of *existence*, and transformed the sentence 'it lightens' into 'the lightning is.' It is only the transformation itself which seems to me unnatural; we never express

[1] ['Wahrnehmung.']

ourselves in this way; the unsophisticated mind does not think of the phenomenon as if it were already something before it existed, of which we could speak, and of which among other things we could assert reality; on the contrary, it regards the particular reality in question as a phenomenon, a predicate, a consequence, proceeding along with others from an antecedent and permanent, though quite inexpressible, subject. Though however we cannot accept this explanation, it is so far right as that every genuine impersonal judgment expresses an actually present perception, and is therefore as regards its form an *assertorial* judgment. Such genuine judgments are to be distinguished from other modes of expression which begin with the indefinite 'it' as subject, but immediately fix its content by an explanatory sentence, as, e.g. 'it is well that this or that should be done.'

50. The more definitely the mind emphasizes the necessity of the subject to which the predicate is to attach, the less can it rest content with an expression in which this demand is unsatisfied. It is not part of my logical task, as I have already said, to describe the processes of comparison and observation by which our ideas of those subjects are gradually formed, which we require to take the places of the indefinite 'it' in the various impersonal judgments; I have only to point out the logical form in which this requirement is satisfied. Most of the simple instances with which logic usually begins its illustration of the judgment in general, are in the familiar form of the categorical judgment '*S* is *P*,' e.g. 'gold is heavy,' 'the tree is green,' 'the day is windy.' No explanation is needed as regards this form; its structure is perfectly transparent and simple: all that we have to show is, that this apparent clearness conceals a complete enigma, and that the obscurity in which the sense of the copula in the categorical judgment is involved will form a motive that will carry us a long way in our successive modifications of logical activity.

51. A certain embarrassment is at once observable as

soon as we ask in what sense *S* and *P* are connected in the categorical as distinct from the hypothetical and disjunctive judgments. A common answer is, that the categorical judgment asserts *S* of *P* *absolutely*; but this answer is only negatively satisfactory, i. e. so far as it denies of the categorical sentence the idea of a condition and the idea of an opposition between mutually exclusive predicates; but when we know what this form of judgment does not do, the statement that it joins *P* to *S* absolutely gives us no positive information as to what it does do. Such a statement in fact merely expresses the greater simplicity of the categorical copula as compared with that of the hypothetical and disjunctive judgments; but this simpler connexion must still have a determinate and expressible sense of its own, distinguishing it from other conceivable forms of connexion equally simple or more complicated. The necessity of explaining this sense appears most simply from the fact, that, of all connexions of *S* and *P*, the complete identity of the two would be that which most obviously deserved the name of absolute. Yet it is just this which as a rule is *not* intended in the categorical judgment: 'gold is heavy' does not mean that gold and weight are identical; equally little do such sentences as 'the tree is green,' 'the sky is blue,' identify the tree with green and the sky with blue. On the contrary, we are at pains to express our real meaning in such judgments by saying, '*P* is not *S* itself, but only a predicate of *S*,' or '*S* *is* not *P*, it only *has* *P*.' We thus admit that we are thinking of a definite and distinguishable relationship between *S* and *P*, and it only remains to make really clear what constitutes this 'having' which we oppose to 'being,' or, in more logical language, wherein we have to look for that relation of a *subject* to its *predicate* which we wish to distinguish from the relation of identity.

52. Plato was the first to touch this problem; his doctrine, that things owe their properties to participation

in the eternal universal concepts of those properties, was rather an inadequate answer to a metaphysical question about the structure of reality, than an explanation of what we have in our mind when we establish a logical relation between subject and predicate. Aristotle made the right treatment of the question possible by observing that the attributes are primarily enunciated of their subjects; this at any rate established the fact that it is a logical operation of the mind which refers the matter of the one concept to that of the other; but more than this name of enunciation, κατηγορεῖν, from which that of the 'categorical' judgment and that of the Latin equivalent 'predicate' are derived, even Aristotle did not discover. He escaped indeed a confusion of later logic; he did not reduce the connexion which he supposed between *S* and *P* from a logical operation to a mere psychical occurrence, thus making the relation between the two consist only in the fact that the idea of *P* is associated in our consciousness with that of *S*: for him the sense of the judgment and the ground for making it was a real relation between the matters of the two ideas. But he did not tell us how precisely *S* is affected by the fact that we enunciate *P* of it; he made the enunciation itself, which can really do nothing but recognise and express this real relation, stand for the very relation which it had to recognise. Now it is easy to see that this fusion is quite inadmissible; it is impossible merely to enunciate the concept 'slave' of Socrates in such a way that the enunciation itself should settle the relationship in which the two concepts stand to one another: what we really mean by a judgment is always, that Socrates is or is not a slave, has or has not slaves, liberates or does not liberate slaves. It is one or other of these possible relations which constitutes *what* is enunciated in each case, and it is only a matter of linguistic usage that, when we speak of enunciating the latter concept of the former, we choose tacitly to under-

stand only the first relation, viz. that Socrates *is* a slave. The relation, therefore, of S to P in a categorical judgment is not distinguished from other relations by saying that P is enunciated of S; the truth rather is that the meaning of this enunciation, in itself manifold, is determined by the tacit supposition that P is enunciated of S as *a predicate of a subject.* It still remains a further question, what constitutes this peculiar relation.

53. We moderns are accustomed on this point to hold to the doctrine of Kant, who represented the relation of a *thing* to its *property*, or of substance to its accident, as the model upon which the mind connects S and P in the categorical judgment. This statement may have a good meaning in the connexion in which Kant made it, but it does not seem to be available for the logical question before us. I will not here raise the point whether the idea of the relation between substance and attribute is itself so clear and intelligible as to dissipate all obscurity from the categorical judgment; it is enough to remind ourselves that logical judgments do not speak only of what is real, of things; many of them have for their subject a mere matter of thought, something unreal, or even impossible. The relation existing between the real thing as such and its properties obviously cannot be transferred in its full sense to the relation of subjects to their predicates, but only in the metaphorical or, as we may say, symbolical sense. To speak more exactly, the only common element in these two kinds of relation is the formal one, that in both the one of the related members, thing, or subject, is apprehended as independent, the other, property or predicate, as dependent upon the former in the way of attachment or inherence. But in regard to the thing, metaphysic has at any rate exerted itself to show how there can be properties which are not the thing and yet attach to it, and what we are to suppose this attachment to consist in; whereas in regard to the relation between

subject and predicate we find no corresponding account of the sense in which the one inheres in the other. The appeal to the relation between thing and property, therefore, does not help logic at all; the question repeats itself, How much of this *metaphysical* relation survives as a *logical* relation expressible in the categorical judgment, if the thing be replaced by something which is not a thing, and the property by something which is not a property?

54. Without adding any more to these customary but unsuccessful attempts to justify the categorical judgment, I will state the conclusion to which we are driven: this absolute connexion of two concepts S and P, in which the one is unconditionally the other and yet both stand over against each other as different, is a relation quite impracticable in thought; by means of *this* copula, the simple 'is' of the categorical judgment, two different contents cannot be connected at all; they must either fall entirely within one another, or they must remain entirely separate, and the impossible judgment 'S is P' resolves itself into the three others, 'S is S,' 'P is P,' 'S is not P.' We must not stumble too much at the startling character of this assertion. Our minds are so constantly making categorical judgments of the form 'S is P,' that no doubt what we *mean* by them will eventually justify itself, and we shall soon see how this is possible. But the categorical judgment *requires* such a justification; taken just as it stands it is a contradictory and self-destructive form of expression, in which the mind either represents as solved a hitherto unsolved problem, the determination of the relation between S and P, or so abbreviates the discovered solution that their connexion is no longer visible. On the other hand we are met by the consciousness that all our thought is subject to a limitation or has to conform to a law; by the conviction that in the categorical judgment each constituent can only be conceived as self-same. This primary law of thought, the *principle of identity*, we express positively in the formula $A = A$, while

in the negative formula, A does not $=$ non-A, it appears as the *principle of contradiction* to every attempt to make $A = B$.

55. I will not interrupt my exposition here by remarks which would have to be repeated later upon the various interpretations which this first law of thought has received; I will confine myself to stating exactly what sense I shall myself attribute to it in opposition to many of those interpretations. In the case of an ultimate principle, which limits the whole of our thinking, it is obvious that with the application of thought to different groups of objects it will be transformed into a number of special principles, which exhibit its general import in the particular forms in which it applies to the particular characteristics of those groups and has an important bearing upon them. The consequences thus drawn from the principle of identity, some of which are quite unexceptionable while others are by no means so, must be distinguished from the original sense of the principle itself, and do not belong to this part of logic. Thus it is quite useless to expand the expression of the law into the formula, Everything can have at the same moment and in the same part of its whole self only one predicate A, and cannot have at the same time a predicate non-A contrary or contradictory to A. This statement is certainly correct, but it is no more than a particular application of the principle to subjects which have the reality of things, are composed of parts, and are capable of change in time. On the other hand it is incorrect to distinguish, as is often done tacitly and not less often explicitly in formulating this principle, between *consistent* predicates, which can belong at the same time to the same subject, and others which cannot because they are inconsistent with one another and with the nature of the subject. In the applications of thought, of course, this distinction too has its validity, when it has justified itself before the law of identity; but, taken as it stands, that law knows nothing of predicates which, though different

from S, are still so far consistent with it that they could be combined with it in a categorical judgment; on the contrary, *every* predicate P which differs in any way whatever from S, however friendly to S it might otherwise be conceived to be, is entirely irreconcileable with it; *every* judgment of the form, 'S is P,' is impossible, and in the strictest sense we cannot get further than saying, 'S is S' and 'P is P.' The same interpretation of the principle must also be maintained against other metaphysical inferences which are drawn from it. It may be that in the course of metaphysical enquiry it becomes necessary to make such assertions as, What is contradictory cannot be real, What is must be unchangeable, and the like: but the logical law of identity says only, What is contradictory is contradictory, What is is, What is changeable is changeable: all such judgments as make one of these concepts the predicate of another require a further special explanation.

B. *The Particular Judgment. The Hypothetical Judgment. The Principle of sufficient Reason.*

56. It would be wearisome to stay longer at a point of view in which we could never permanently rest: we will follow thought to the new forms in which it tries to bring its categorical judgments into harmony with the law of identity. Judgments of the form 'S is P' are called synthetical, when P is understood to be a mark not already contained in that group of marks which enables us to conceive S distinctly; they are called analytical when P, though not identical with the whole of S, yet belongs essentially to those marks the union of which is necessary to make the concept of S complete. In the analytical judgment people have found no difficulty; but the synthetical attracted attention at an early period, and Kant's treatment of it in particular has recently made it conspicuous. He too however was mainly interested in accounting for the possibility of synthetical judgments *a priori*,

i.e. such as assert an existing and necessary connexion between S and a concept P not indispensable to S, without the need of appealing to the experience of its actual occurrence: as to synthetic judgments *a posteriori*, which merely state that such a connexion between two not mutually indispensable concepts is found or has been found in experience, he regarded them as simple expressions of facts and therefore free from difficulty. These distinctions may be fully justified within the circle of enquiry in which Kant moved; but our logical question as to the possibility of categorical judgments extends to all three forms with equal urgency. The necessity of justification before the principle of identity is only more obvious in the case of the *a priori* synthetical judgment, which formally contradicts that principle; but it holds good of the *a posteriori* also. For a judgment does not simply reproduce the fact like a mirror; it always introduces into the observed elements of the fact the thought of an inner relation, which is not included in the observation. Experience shows us only that S and P *are* together; but that they are inwardly *connected*, as we imply when we predicate P of S in the judgment, is only the interpretation which our mind puts upon the fact. How this relation can subsist between subject and predicate in general, and between S and P in particular, is just as obscure after experience has shown the coexistence to be a fact as when we assert it in anticipation of experience. Lastly, analytical judgments raise the same difficulty. However much yellow may be already contained in the concept of gold, the judgment 'gold is yellow' does not assert merely that the idea of yellow lies in the idea of gold, but ascribes yellowness to gold as its property; gold must therefore have a determinate relation to it, which is not the relation of identity. This relation has to be explained, and the question still remains, What right have we to assign to S a P which *is* not S, as a predicate in a categorical judgment?

57. The only answer can be, that we have no right: the numberless categorical judgments of this form which we make in daily life can only be justified by showing that they *mean* something quite different from what they say, and that, if we emphasize what they mean, they are in fact identical judgments in the full sense required by the principle of identity. The first form in which we get a hint of this in the natural course of thought is that of *quantitative* judgments in general, which I shall in future call shortly *particular*, and consider as the first form of this second group of judgments. Under this title I include not only the traditional forms, such as, 'all *S* are *P*,' 'some *S* are *P*,' this '*S* is *P*,' which have for their subject a number of instances of the general concept *S*, but those also which in various other ways limit to definite cases, and therefore particularise, the universal application of the connexion between *S* and *P*, whether by particles of time (now, often, etc.), or by those of space (here, there, etc.), or again by a past or future tense of the verb, or lastly by any kind of accessory idea, imperfectly expressed or not expressed at all. In the general formula of the categorical judgment, '*S* is *P*,' it looks as if the universal *S* were the subject, the universal *P* its predicate, and the constant, unchangeable, and unlimited connexion of *S* and *P* the import of the whole judgment. If on the other hand we supply explicitly what is suggested, or at any rate is meant, by these particularising accessory ideas, we find that the true subject is not the universal *S*, but Σ, a determinate instance of it; that the true predicate is not the universal *P*, but Π, a particular modification of it; and lastly that the relation asserted is not between *S* and *P*, but between Σ and Π, and that this relation, if the supplementary ideas are correct, is no longer a synthetical, nor even an analytical one, but simply one of identity. A few instances will make this clear.

58. We say, 'some men are black,' and suppose ourselves

to be making a synthetical judgment, because blackness is not contained in the concept of man. But the true subject of this sentence is not the universal concept 'man' (for it is not that which is black), but certain individual men; these individuals, however, though they are *expressed* as merely an indefinite portion of the whole of humanity, are yet by no means *understood* to be such an indefinite portion; for it is not left to our choice what individuals we will take out of the whole mass of men; our selection, which makes them 'some' men, does not make them black if they are not so without it; we have, then, to choose those men, and we *mean* all along only those men, who are black, in short, negroes; these are the true subject of the judgment. That the predicate is not meant in its universality, that on the contrary only the particular black is meant which is found on human bodies, is at once clear, and I shall follow out this remark later; here I will only observe that it is merely want of inflexion in the German expression which deceives us as to its proper sense; the Latin 'nonnulli homines sunt nigri' shows at once by number and gender that 'homines' has to be supplied to 'nigri.' The full sense, then, of the judgment is, 'some men, by whom however we are only to understand black men, are black men'; as regards its matter it is perfectly identical, and as regards its form it is only synthetical because one and the same subject is expressed from two different points of view, as black men in the predicate, as a fragment of all men in the subject. Again, we say, 'the dog drinks.' But the universal dog does not drink; only a single definite dog, or many, or all single dogs, are the subject of the sentence. In the predicate too we mean something different from what we express: we do not think of the dog as a sort of ever-running syphon; he does not drink simply, always, and unceasingly, but now and then. And this 'now and then' also, though expressed as an indefinite number of moments, is not so meant; the dog drinks only at definite moments, when he is thirsty or at

any rate inclined, when he finds something to drink, when nobody stops him; in short, the dog which we mean in this judgment is really only the drinking dog, and the same drinking dog is also the predicate. Again, 'Caesar crossed the Rubicon'; but not the Caesar who lay in the cradle, or was asleep, or was undecided what to do, but the Caesar who came out of Gaul, who was awake, conscious of the situation, and had made up his mind; in a word, the Caesar whom the subject of this judgment means is that Caesar only whom the predicate characterises, the Caesar who is crossing the Rubicon, and in no previous moment of his life was he the subject to whom this predicate could have been attached. It is obvious moreover to every capacity that when he had crossed the river he could not go on crossing it, but was across, so that in no subsequent moment of his life either can he be the subject intended in this judgment. I will give two more examples, which Kant has made famous. It is said that the judgment, 'a straight line is the shortest way between two points,' is synthetical, for neither in the concept 'straight' nor in that of 'line' is there any suggestion of longitudinal measure. But the actual geometrical judgment does not say of a straight line in general that it is this shortest way, but only of that one which is included between those two points. Now this fact, the fact that its extension is bounded by two points, (and it is only with this qualification that it forms the true subject of the sentence) is the ground, in this case certainly the satisfactory ground, for assigning the predicate to it. It is easy to see that the concept of a straight line $a\ b$ between the points a and b is perfectly identical with the concept of the distance of the two points; for we cannot give any other idea of what we mean by 'distance in space' than this, that it is the length of the straight line between a and b. There is not therefore a shorter and a longer distance between a and b, but only the one distance $a\ b$, which is always the same. On the other hand, we can speak of

shorter and longer *ways* between *a* and *b*; the concept of way implies merely any sort of progression which leads from *a* to *b*; as this requires the getting over of the difference which separates *b* from *a*, there can be no way leading from *a* to *b* which leaves any part of this difference not got over; accordingly, that the shortest of all possible ways is the distance, i.e. the straight line between the given points, is a judgment which, as regards its matter, is perfectly identical, and merely regards the same object from different aspects. Nor again can the arithmetical judgment, $7+5=12$, be synthetical because 12 is not contained in either 7 or 5: the complete subject does not consist in either of these quantities singly, but in the combination of them required by the sign of addition; but in this combination, if the equation is correct, the predicate must be wholly contained; the equation would be false if some unknown quantity had to be added to $7+5$ in order to produce 12. Here too, then, we have a perfectly identical judgment as regards its matter, and it is only synthetical formally because it exhibits the same number 12 first as the sum of two other quantities, and then as determined by its order in the simple series of numbers. I must now add that it is impossible to express everything satisfactorily all at once: what it really means, and how it is possible, that thought should represent the same matter under different forms, we shall very soon have occasion to consider; and subsequently it will appear that my late remarks were not intended to charge Kant with a logical oversight so easily detected.

59. So far our result seems to be this: categorical judgments of the form '*S* is *P*' are admissible in practice because they are always conceived in the sense which we have called particular, and as such are ultimately identical. No one however will feel satisfied with this conclusion: it will be rightly objected that it does away with the essential character of a judgment, which is that it expresses a coherence between the contents of two ideas.

In fact, if, by the supplementary additions which we spoke of, we make our examples into identical judgments, and thus compress their whole content into their subjects, so that A means the black man, B the drinking dog, C Caesar crossing the Rubicon, all that they say, except the barren truth that $A = A$, $B = B$, $C = C$, is reduced to this, that A exists as a fact continually, B sometimes, and C has occurred once in history. In other words, these judgments no longer assert any *mutual relation* between the parts of their *content*, but only that this content as a composite whole is a more or less widely extended *fact*, and this is clearly a relapse to the imperfect stage of the impersonal judgment. The following consideration will make us still more sensible of this defect. I just now described B as the concept of the drinking dog, but properly I had no right to do so; for this expression, which joins 'drinking' in the form of a participle to the subject 'dog,' is itself only conceivable and admissible on the assumption that the mark of drinking, P, which is not contained in the subject S, can really be ascribed to that subject in a categorical judgment, and ascribed to it in the sense of its property or state. Now just this possibility has been done away with by our previous explanation; all that it is now competent to us to do is to understand B as the coexistent sum of its marks $a\ b\ c\ d$, and to say, this $a\ b\ c\ d$, which according to the principle of identity is always self-same, has a certain reality, while another aggregate of marks, $a\ b\ c\ e$, has a similar reality on another occasion. But we have no right whatever to regard the common group a, b, c, as something inwardly connected, and more connected in itself than with the varying elements d and e, still less as something which offers a support to these changing elements as subject to attributes. In language, indeed, we should continue to describe this $a\ b\ c$ as 'dog,' $a\ b\ c\ d$ as 'eating,' and $a\ b\ c\ e$ as 'drinking dog'; but these expressions would rest upon no logical ground; none of our judgments could express any-

thing but simple or composite perceptions, and between the several perceptions, or even the several parts of each composite perception, there could be no expressible connexion such as could show their mere coexistence to be due to inner coherence.

60. Against such a complete failure in its logical purpose the mind guards itself, by further transforming the particular judgment in a way which may be primarily considered as a simple denial that the material of our ideas is thus disintegrated into merely isolated coexistent facts. The additions by which we supplemented the subject S expressed in the categorical judgment, were the means by which we helped that judgment to justify itself before the principle of identity; they are now recognised as being also the valid ground of fact which qualifies S for assuming a predicate P, which, so long as it stood alone, would not belong to it. The accessory circumstances, through which S first became the true subject Σ of a then identical judgment, appear now as the *conditions*, by the operation or presence of which S is so influenced that a P, which before was strange to it, now fits and belongs to it consistently with the principle of identity. It is therefore the *hypothetical* judgment which takes its place as the second member in this second group of the forms of judgment; it is compounded of a protasis and an apodosis, which in the simplest typical case have the same subject S, but different predicates, in the protasis a Q which expresses the condition accruing to S, in the apodosis a P which expresses the mark produced in S by that condition. All hypothetical judgments with different subjects in their two members are abbreviated expressions, and can be reduced by easily supplied links to this original form, 'If S is Q, S is P.' If it is further wished to imply that the protasis, which as such is only problematical, is actually true, we get the form, '*Because* S is Q, S is P': and lastly, the assertion that Q is *not* the ground for S's being P gives rise to the last form which we need mention,

'*Although* S is Q, yet S is not P.' Logically there is nothing peculiar in these two forms.

61. This short survey is quite sufficient to characterise the external forms of the hypothetical judgment. But an observant reader must ask at this point, what right had we to translate those supplementary additions, to which the true subject Σ of the then identical judgment owed its origin, into *conditions*, which, by operating upon an already existing subject S, *give a ground for* the predication of P. The principle of identity merely asserts the sameness of everything with itself; the only relation in which it places two different things is that of mutual exclusion. If then we supposed various simple elements $a\ b\ c\ p\ q$ existing together in some real form, but without being in any way inwardly connected, some of these elements might equally well occur at any subsequent moment in any other combination with any other element, and the fact of our observing $a\ b\ c\ q$ a second time would not enable us to conclude that p must be there too; any r or s might with equal right take its place. On the other hand, if we make the quite general presupposition that the totality of things thinkable and real is not merely a sum which coexists but a whole which coheres, then the law of identity has wider consequences. The *same* $a\ b\ c\ q$, with which p has once been found in combination, can then according to the law of identity never be found in combination with a non-p, nor can *this* $abcq$ ever occur without its former predicate p. *How* such a cohesion between different elements is conceivable, we will leave for a moment an open question; but *if* it exists, it must exist in an identical form in every recurrent instance, and (confining ourselves to a combination of three elements) given $a\ b$, c is the only new element which can necessarily accrue, given $a\ c$, b, and given $b\ c$, a; in other words, whichever of these elements occurs first in any case has in the second the sufficient and necessary condition for the possibility and necessity of the accession of the third. That

element or group of elements to which we here give the first place, appears to us then logically as the subject; that which we place second, as the condition which operates upon this subject, while the third represents the consequence produced in the subject by the condition. I wish further expressly to point out that this choice of places is quite arbitrary, and in practice is decided by the nature of the object and our interest in it: in itself, every element in such a combination is a function of the rest, and we can pass inferentially from any one to any other. It is usual to conceive of a number of elements which frequently recur together as a subject S, which generally signifies a thing or permanent reality: on the other hand, a single element b, which is absent in some observations of S and present in others, is conceived as the accessory condition Q, and $a\ c$, which always accompanies b, as the consequence P of which Q is the condition. But it is obvious that we may proceed in a different way; and in fact mechanical physics are able to treat the single and uniform force of gravity, b or Q, as a subject, and to investigate the various consequences, P, which accrue to it if the bodies upon which it acts ($a\ m\ n = S$ or $a\ m\ r = S^{1}$) be regarded as different conditions to whose influence it is liable.

62. In this way the interpretation by which we arrived at hypothetical judgments may be said to be so far justified, as that it has been traced back to the most general assumption of a coherence between the various contents of thought. To prove further than this the admissibility and truth of that assumption itself, cannot be part of our undertaking; any such attempt would obviously imply what had to be proved, for how could we show that it is permissible and necessary to conceive the matter of experience as a web of *reasons* and *consequences*, if we did not base this assertion itself upon a reason of which it was the consequence? This idea of the coherence of the world of thought must therefore either be apprehended with immediate certitude, as the

soul of all thinking, or we must give it up and along with it everything that depends upon it. On the other hand, we are justified in desiring further elucidation of the possibility and the meaning of such a coherence of different elements. The possibility of mutual relations between what is different is not really threatened by the principle of identity, according to which each thing is related only to itself; for all that this principle can affirm is the content of the thing itself; it cannot exclude other contents which do not conflict with it. But as regards the meaning of the coherence, we must distinguish two questions. In logic as here conceived we do not trouble ourselves at all as to what the real process may be through which the unknown reality, which we express well or ill through our ideas, reacts upon itself and produces changes in its conditions; to reflect upon the bond of this connexion is the function of metaphysic, and the question should find solution in a theory of the efficient cause. Logic, on the other hand, which includes in its consideration the relations of the merely thinkable which has no real existence in fact, is confined to developing the other principle, that of *sufficient reason*; it has merely to show how, from the combination of two contents of thought, S and Q, the necessity arises of *thinking* a third, P, and this in a definite relation to S; if then it were found in actual experience that such a union of S^1 and Q^1 is an accomplished fact, the particular consequence P^1, which according to the necessity of thought must follow such a combination in distinction from P^2 which could not so follow, could be inferred according to the principle of sufficient reason; but how it comes about that the very P^1, which is required by thought, occurs in reality as well, is a question which would be left to the metaphysical enquiries referred to.

63. The *law of sufficient reason*, with which we now conclude as the third member and the net result of this second group of the forms of judgment, much as it has been talked about, has had the curious fortune never to have been, pro-

perly speaking, formulated, even by those who most frequently appealed to it. For the ordinary injunction, that for every statement which claims validity we must seek a ground for its validity, forgets that we cannot seek for that of which we do not know wherein it consists; clearly the first thing that has to be explained is, in what relation reason and consequence stand to each other, and in what sort of thing consequently we may hope to discover the reason of another thing. I shall make my meaning clear in the shortest way, if, on the analogy of the expression of the principle of identity, $A = A$, I at once give the formula $A + B = C$ as the expression of the principle of sufficient reason, adding the following explanation. Taken by themselves, A only $= A$, $B = B$; but there is no reason why a particular combination $A + B$, the very different sense of which in different cases is here represented by the sign of addition, should not be equivalent to, or identical with, the simple content of the new concept C. If we thus call $A + B$ the reason and C the consequence, reason and consequence are completely identical, and the one *is* the other; in this case we must understand by $A + B$ any given subject along with the condition by which it is influenced, and by C, not a new predicate which is the consequence of this subject, but the subject itself in its form as altered by the predicate. In ordinary usage this is expressed differently. Inasmuch as, in speaking of real facts, the one part A is usually already given, while the other B is a subsequent addition, it is customary to describe the *condition* B, which forms only a part of the whole reason $A + B$, as the reason in general which acts upon the passive subject A; by C is then usually understood nothing but the new property conditioned by B, and this is called the consequence; at the same time, however, the property is never thought of as existing on its own account, as if in empty space, but as attaching to the subject A upon which B was supposed to act. Ordinary usage, therefore,

though it employs a different nomenclature, means the same as I do. If with the idea of powder, A, we connect the idea of the high temperature of the spark, B, and thus substitute B for the mark of ordinary temperature in A, then $A + B$ really *is* the idea C of exploding powder, not of explosion in general; the ordinary usage makes the high temperature B seem to supervene on the given subject A as a reason from which the explosion C follows; but of course it conceives this consequence, not as a process which takes place anywhere, but as an expansion of the particular powder upon which the spark acted. It is not necessary to continue such simple explanations any further.

64. If we consider the whole of our knowledge, we see at once that the principle of identity cannot be its only source. Taken alone it would isolate every judgment and even every concept, and would not open any way to a progress from the barren self-identity of single elements of thought to their fruitful combination with others. It is a mistake, as is sometimes done, to represent this single principle as the basis of the truths of mathematics; the fact is that here too it is only the principle of sufficient reason which helps to real discovery. From a self-identical major premiss nothing new could flow, unless it were possible in a number of minor premisses to give the same quantity C innumerable equivalent forms, at one time $= A + B$, at another $= M + N$, at another $= N - R$; or, to express the same thing otherwise, unless the nature of numbers were such that we can divide them all in innumerable ways and compound them again in the most manifold combinations; and again, unless the nature of space were such that every line can be inserted as a part or otherwise coherent member in innumerable figures in the most various positions, and that each one of the expressions for it, which flow from these various relations, is the ground for new and manifold consequences. I need hardly mention that mechanics and physics also make the most

extensive use of this analysis and composition of given facts, and that the process of thought in discovery in these branches of knowledge rests upon operations which all ultimately come back to the typical formula, $A + B = C$. To Herbart belongs the credit of having brought within the ken of formal logic the importance of a mode of procedure so prominent in all scientific practice.

65. Reserving further illustrations for applied logic, I have another remark to make about the justification of the principle of sufficient reason itself. We were only able to show that an extension of our knowledge is possible *if* there is a principle which allows us to make $A + B = C$. We might accordingly attempt to assert the validity of this principle at once, as an immediate certitude, like the principle of identity. This is what we have done; still there is a noticeable difference between the two principles. The principle of identity expresses of every A an equality with itself which we feel immediately to be necessary, and the opposite of which also we feel with equal conviction to be impossible in thought. The principle of sufficient reason lacks this latter support; we do not by any means feel it impossible to suppose that, while every content of thought is self-identical, no combination of two contents is ever equivalent to a third. The validity of the latter principle, therefore, is of a different kind from that of the former; if we call the one necessary to thought because of the impossibility of its opposite, the other must be considered rather as an assumption which serves the purposes of thought, an assumption of mutual relatedness in thinkable matter the truth of which is guaranteed by the concentrated impression of all experience.

I wish not to be misunderstood in this last phrase. In the first place, I do not mean that it is a comparison of what we experience which first leads the mind to conjecture the validity of such a principle; the general tendency of the logical spirit, to exhibit the coexistent as coherent,

contains in itself the impulse, which, independently even of all actual experience, would lead to the assumption of a connexion of reasons and consequences. But that this assumption is confirmed, that thought does come upon such identities or equivalences between different elements in the thinkable matter which it does not make, but receives or finds, this is a fortunate fact, a fortunate trait in the organisation of the thinkable world, a trait which does really exist, but has not the same necessity for existing as the principle of identity. It is not impossible to conceive a world in which everything should be as incommensurable with every other thing as sweet is with triangular, and in which therefore there was no possibility of so holding two different things together as to give ground for a third: it is true that, if such a world existed, the mind would not know what to do with it, but it would be obliged to recognise it as possible according to its own judgment. I will add further that, when I speak of a kind of empirical confirmation of the principle of sufficient reason, I do not mean such a confirmation as the whole of our world of thought, already articulated in accordance with that principle, might find in the fact that *external* reality, so far as it is observable, corresponds with this articulation; I am speaking here only of the fact that the thinkable world, the contents of our ideas which, whatever their source, we find in our *inner* experience, do really conform to the requirement that they should cohere as reasons and consequences. In this stage of logic it is quite indifferent whether or not there is anything which can be called external world or reality besides the ideas which move within our consciousness; like that reality, this internal world itself, with all that it contains, is not made by thought; it is a material which thought finds in us to work upon, and it is therefore for the logical spirit and its tendency an object of inner experience; *this*, then, is the empirical object which, by responding to the logical

tendency and making its realisation possible, substantiates the principle of sufficient reason, not as a necessity of thought, but as a fact.

66. As to the nature of this responsiveness in the world of thought (if that question is to be raised again here), the shortest way to recall it is to observe that the position occupied in the system by the principle of sufficient reason, as the second law of thought, is analogous to that of the act which[1] we placed second in treating of conception. The possibility of forming general concepts depended on the fact, not in itself a necessity of thought, that every idea is not incommensurable with every other, but that, on the contrary, colours, tones, and shapes group themselves in series of cognisable gradations; that further there are oppositions of varying degree, as well as affinities, in the world of thought, and that opposites cancel one another; and lastly, and most important of all, that there is a system of quantitative determinations enabling us to compare the members of different series, which as such stand in no mutual relation. With this brief indication, we leave the principle of sufficient reason as the conclusion and net result of the second group of the forms of judgment.

C. *The General Judgment. The Disjunctive Judgment. The* Dictum de omni et nullo *and the* Principium exclusi medii.

67. It remains to determine in each particular case, What *A*, combined in what form with what *B*, forms the adequate reason of what *C*. This question of fact logic must leave to experience and the special sciences; but a new question is developed which logic itself must deal with. There would be little result from all the activity of our mind if we were really obliged in *every* particular case to renew the question to experience, What *A*, *B*, and

[1] [See above, § 19.]

C in this instance cohere as reason and consequence? There must be at any rate *a* principle which allows us, when once the one truth $A+B=C$ is given, to apply it to cases of which experience has not yet informed us. What we are here looking for is easy to find, and has been already mentioned incidentally. Whenever we regard $A+B$ as the reason of a consequence C, we necessarily conceive the connexion of the three as a *universal* one; $A + B$ would not be a condition of C, if, in a second case of its occurrence, some casual D instead of C might possibly be found combined with it. The significance of this in its present application is as follows: everywhere, in every subject S in which $A + B$ is contained as a mark along with other marks, $N\ O\ P$, this $A+B$ gives ground for the same consequence C; and this C will either actually occur as a mark of S, or, if it does not occur, it can only be hindered because the other marks, $N+O$ or $N+P$ or $O+P$, formed together the ground for a consequence opposed to and destructive of C; taken by itself, without this hindrance, the power of $A+B$ to condition C never loses its effect. If now we conceive $A+B$ under the title M as a universal concept to which S is subordinate, we may give the following preliminary expression to the principle just discovered, viz. that by right of pure logic and without appeal to experience every subject may have that predicate affirmed of it which is required by the generic concept above it. And it is clear without further explanation that this very idea of the subordination of the individual to the universal is the comprehensive logical instrument, of which we avail ourselves whenever we want to carry further the work of thought upon the material given in experience.

68. The form of judgment, the first of this third group, in which the mind expresses this conviction, is that of the quantitatively undetermined proposition, in which the place of the subject is filled simply by a universal or generic concept M; 'man is mortal,' 'sin is punishable.' I dis-

tinguish these as *general* judgments from the universal ones, 'all men are mortal,' 'every sin is punishable.' Although the fact contained in both forms is the same, the logical setting of it in the two cases is quite different. The universal judgment is only a collection of many singular judgments, the sum of whose subjects does as a matter of fact fill up the whole extent of the universal concept; thus the fact that the predicate P holds good of all M follows here only from the fact that it holds good of every single M; it may however hold good of each M for a special reason which has nothing to do with the universal nature of M. Thus the universal proposition, 'all inhabitants of this town are poor,' leaves it quite uncertain whether each inhabitant is made poor by a particular cause, or whether the poverty arises from his being an inhabitant of this town; so too the universal proposition, 'all men are mortal,' leaves it still an open question whether, strictly speaking, they *might* not all live for ever, and whether it is not merely a remarkable concatenation of circumstances, different in every different case, which finally results in the fact that no one remains alive. The general judgment on the other hand, 'man is mortal,' asserts by its form that it lies in the character of mankind that mortality is inseparable from every one who partakes in it. While therefore the universal judgment merely states a universal fact, and is therefore only assertorial, the general judgment lets the reason of its necessary truth be seen through it, and may thus, in the sense laid down above[1], be called apodeictic. This distinction of the two forms of judgment will not lead to any unheard-of discoveries; but in comparison with the many unprofitable distinctions which encumber logic it deserved an incidental mention. It is scarcely necessary to remark that in the general judgment it is not the generic concept M, occupying the place of subject in the sentence,

[1] [Above, § 42.]

which is the true logical subject of the judgment; it is not the universal man who is mortal, but the individual S who participates in this type, which in itself is immortal. From this we see that the general judgment is properly an abbreviated hypothetical judgment; in its full form it ought to stand, 'If S is M, S is P,' 'If any S is a man, this S is mortal.' And this justifies us in not introducing it in our system until after the hypothetical.

69. But it is no less clear that we must make another step. So long as a universal generic concept M occurs as formally the subject in the general judgment, the predicate P which is joined to it can only be understood with equal universality. If we say, 'man is mortal,' the predicate embraces all conceivable kinds of mortality, and does not determine either the manner or the moment of death; or if we say, 'bodies occupy space,' it remains unexpressed with what degree of density and of resistance each single body realises the universal property of its class. But we saw that it is individual men and individual bodies which are the real subjects of the general judgment; it is therefore quite false to say that P, the mark of their class, is a predicate of the individuals in the same universal sense in which it is joined in thought (and that not as a predicate) to the concept of the class; the truth is that P can only occur in each one of these individuals in one of the definite forms or modifications into which the universal P can be analysed or particularised. The mind corrects this mistake by means of the fresh assertion, 'If any S is an M, this S is either p^1 or p^2 or p^3,' and here p^1 p^2 p^3 mean the different kinds of a universal mark P which is contained in the generic concept M. This is the familiar form of the *disjunctive judgment*, the second in this third group, and one which, as such, requires no further explanation. It is usual to mention along with it the *copulative* judgment ('S is both p and q and r'), and the *remotive* judgment ('S is neither p nor q nor r'); but

in spite of the external analogy of form, neither of these has the same logical value as the disjunctive; the first is only a collection of positive, the second of negative, judgments with the same subject and different predicates, which latter are not placed in any logically important relation to each other. The disjunctive judgment alone expresses a special relation between its members: it gives its subject no predicate at all, but prescribes to it the alternative between a definite number of different predicates.

70. The thought expressed by the form of the disjunctive judgment usually finds utterance in two separate laws of thought, the *Dictum de omni et nullo* and the *Principium exclusi tertii inter duo contradictoria*; but the amalgamation of them in a single third law is not only easy but necessary. The careless formulations often given of the first are completely false, e.g. 'What is true of the universal is true also of the particular,' 'What is true of the whole is true also of the parts'; on the contrary, it is self-evident that what holds good of the universal as such or of the whole as such, cannot hold good of the individual as such or of the parts as such. The only correct formula is, *quidquid de omnibus valet valet etiam de quibusdam et de singulis*, and *quidquid de nullo valet nec de quibusdam valet nec de singulis*. But this form of expression (for the history of which see Rehnisch, Fichte's Zeitschrift, lxxvi, 1) is as barren as it is correct; for to hold good of *all* is and means from the very first nothing else than to hold good of each one; if therefore anything worth saying is to take the place of this bare tautology, the nature of the universal concept must certainly be substituted for the mere sum of *all*. But in that case the principle cannot really be accurately expressed except in a form which means precisely the same as the disjunctive judgment; viz. whenever a universal P is a mark in a universal concept M, one of its modifications, $p^1 p^2 p^3$, to the exclusion of the rest, belongs

to every S which is a species of M; whenever a universal P is excluded from a concept M, no one of the modifications of P belongs to any S which is a species of M.

71. Of this complete law of thought the ordinary expression of the *dictum de omni et nullo* only regards the one and positive part, which, as we saw, cannot by itself be accurately expressed, the general idea, namely, that the particular is determined by its universal: the other and negative part, which defines the manner of this determination, the idea that the particular admits only one specific form of its generic predicate to the exclusion of the others, has found only a partial expression in the principle of the excluded middle. I think that I can say what I have to say about this most simply as follows. Suppose a subject S subordinate to M, and that this subordination implies that S must choose its own predicate from amongst p^1 p' p^3, the specific forms of P, a universal mark belonging to M, then, if there are more than two of these forms, the affirmation of one of them as predicate of S will involve the negation of all the rest, but the negation of one of them will not involve the affirmation of any particular one of the rest; what is not p^1 has still an open choice between p^2 p^3 p^4. To predicates of this sort it is usual to ascribe the opposition of *contrariety*. If however there are only two specific forms of P, p^1 and p^2, and S must have a specific form of P for its predicate, then not only does the affirmation of one of them as predicate of S involve the negation of the other, but also the negation of the one involves the definite affirmation of the other; p^1 and p^2 are then opposed to one another *contradictorily*. Thus for the line (S), which must have some direction (P), straight (p^1) and crooked (p^2) are contradictory predicates, and so for man, whose nature it is to have sex, are male and female: for any other subjects, of which it was not yet established whether their concepts contained the universal P at all, these predicates would be only contrary; for such subjects the division of their

possible predicates will be always threefold, they are either male, female, or sexless, either straight, crooked, or formless. Now the principle of the excluded middle asserts nothing but what we have just remarked, that of two predicates which are contradictory for a subject *S*, *S* always has one to the exclusion of the other, and if it has not the one it necessarily has the other to the exclusion of any third. So regarded, this law is only a particular case of the more universal law of which the disjunctive judgment is the expression, viz. that of all contrary predicates whose universal *P* is contained in the generic concept *M* of a subject *S*, *S* has always one to the exclusion of the rest, and if it has not any given one, it has only left it the choice between the others; this choice becomes a definite affirmation when it can only fall on one member, i. e. in the extreme case where the number of contrary predicates is only two. Such a case, which is all that is covered by the principle of the excluded middle, is no doubt of peculiar importance in practice, but a system of logic can only treat it as a particular instance of the more universal principle, which we have already mentioned several times and which we will briefly describe as the *disjunctive law of thought*.

72. It is usual to represent this differently. From motives which are likewise only intelligible on practical grounds, the logical desire has arisen to omit the presupposition to which we have adhered throughout (viz. that the given subject *S* be already understood to stand in a necessary relation to the universal predicate *P*), and to be allowed to speak of two predicates which hold good as contradictories of any subject whatever. It is soon found that this is only possible, if the aggregate of all conceivable predicates be divided into a definite *Q* and the sum of all those which are not *Q* or non-*Q*; it is then certain that any subject, whatever it may mean, must be either *Q* or non-*Q*, either straight or not-straight, for not-straight will include not only

crooked, but annoying, sweet, future, everything in short which lies outside of straight. On this point I may repeat what I said[1] about the limitative judgment, viz. that non-*Q* is not a real idea at all, such as can be treated as predicate of a subject; it is only a formula expressing a mentally *impracticable* task, the collection of all thinkable matter that lies outside a given concept into a single other concept. Moreover there is no real reason for propounding this insoluble problem; everything which it is wished to secure by the affirmative predicate non-*Q* is secured by the intelligible negation of *Q*. I therefore consider it quite improper to speak of contradictory *concepts*, i.e. concepts which are of themselves contradictorily opposed and *therefore* retain that opposition when treated as predicates of one and the same subject, whatever that subject may be: if we want a contradictory relation which shall hold good universally, always, and in regard to every subject, it can only exist between two *judgments*, '*S* is *Q*,' '*S* is not *Q*.' Accordingly the precise expression of the principle of the excluded middle would be, that of every precisely determined subject *S* either the affirmation or the negation of an equally determinate predicate *Q* holds good, and no third alternative is possible; wherever it appears to be possible, *S* or *Q* or both have either been taken in more than one sense or in an indefinite sense in the first instance, or their meaning has been unconsciously or involuntarily changed in the course of reflection.

73. I have one more observation to add. No one doubts that the same subject can be at the same time red, sweet, and heavy, but that it is red only when it is neither green nor blue nor of any other colour, and that it cannot be straight and crooked at the same time. Yet it does not seem to me to be immediately evident that, as is sometimes asserted, the case in which two predicates p^1 and p^2 are incompatible in the same subject is just that in which they

[1] [See above, § 40.]

are contrary species of the same universal P and therefore admit of comparison, whereas other predicates p q r are compatible in the same subject when, as species of quite different universals P Q R, they admit of no comparison. On this point I venture the following reflections. Every predicate p^1 of a subject S must be regarded, in accordance with what we said above and the formula $A+B=C$, as a consequence of a group of marks A^1+B^1 in S, which group tends in all cases (and therefore in the case of S) to produce the same result C^1 (in this case p^1). If now the same S is to have at the same time the predicate p^2, comparable with p^1, it is easy to understand that p^2 must depend on a group of marks A^2+B^2, similarly comparable with A^1+B^1, existing side by side with the latter in S, and in all cases of its occurrence (and therefore in the case of S) giving ground for the result C^2 (in this case $p^{\prime\prime}$). But the consequence of the very comparability of A^1+B^1 and A^2+B^2 must be that, according to a new principle of the general form $A+B=C$, viz. $[A^1+B^1]+[A^2+B^2]=C^3$, their meeting in the same subject S will furnish the sufficient reason for a new consequence C^3, in which the two specific predicates p^1 and p^2 coalesce, and which, as it must resemble both of them, we will call p^3. The only reason, therefore, why two *contrary* and comparable predicates p^1 and p^2 would be irreconcileable, is that they would always give rise to a third and simple p^3; on the other hand, two *disparate* and incomparable predicates p and r, such as sweet and warm, could coexist permanently in S because there is no principle such as $(A+B)+(M+N)=C$ enabling the two disparate grounds $A+B$ and $M+N$, on which the predicates respectively depend, to produce like p^1 and p^2 a third and simple predicate. I will not quarrel with those who find the whole of this exposition superfluous; it seems to me to have some point, when I turn from the examples which logic traditionally employs to others which it would do well not to forget. When anyone says of gold that it is yellow,

he has, it is true, no occasion to think of this simple property as a product of two other imperceptible ones, which properly speaking must have been produced separately by two conditions coexisting in gold, but could not remain separate. But when two motive forces contrary or even contradictory in direction act upon a material point, that which in the previous case would have been a needless assumption is now an actual *fact*; we have to conceive both of the condition which tends to produce the motion p^1 and of that which tends to produce p^2 as operating at the point, and of the two motions themselves as at every moment predicates of that point, but predicates which cannot maintain themselves separately but coalesce in a third p^3, the motion in the diagonal.

And ultimately this is seen to be true in all cases. A crooked line may appear indifferently red or green: but if the conditions of both appearances were operating at the same time and with the same force, it would help us but little to assert, on the principle of exclusion, that the image of the line cannot have these two contrary properties; it *must* present *some* appearance. As however these two conditions are comparable and capable of forming a resultant, a third colour will appear, the production of which will satisfy the claims of the two conditions, but will at the same time contain the reason why the two contrary colours, which singly they would have produced, cannot exist separately side by side.

74. The series of judgments concludes here by an inherent necessity. The more definitely the disjunctive judgment prescribes to its subject the choice between different predicates, the less can this uncertainty be final; the choice must be made. But the decision, *what* p^1 or p^2 belongs to S, cannot come from the fact (which is thus far the only fact) that S is subordinate to M, for it is just because it is a species of M that it is still free to choose: that decision can only come from the specific difference by which S, as *this* species

of M, is distinguished from other species of it. The proposition 'M (and every S which is M) is P,' must therefore have added to it a second proposition which brings to light the specific character of S, the particular subject always in question, and shows us *what* species of M it is; and from the union of the two propositions must arise a third, informing us what particular modification p of the universal P belongs to *this* S because it is, not only a species of M, but *this* species. The form of thought which combines two judgments so as to produce a third is, speaking generally, *inference*, and it is therefore to the exposition of inference that we have now to pass.

Appendix on immediate inferences.

In conformity with tradition I insert some explanations here which would more correctly come under the head of applied logic. Of the same subject S and the same predicate P the universal affirmative judgment, A, asserts 'All S are P,' the particular affirmative, I, 'Some S are P,' the universal negative, E, 'No S is P,' and the particular negative, O, 'Some S are not P.' The question is, what *immediate* inferences can be drawn from the truth or untruth of one of these four judgments in regard to the truth or untruth of the other three? From the *Dictum de omni et nullo* and the principle of the excluded middle we obtain the following results.

75. Between each universal judgment and the particular of like name there is the relation of *subalternation*. Going from the universal to the particular or *ad subalternatam*, we infer the truth of the latter from that of the former, but from the untruth of the universal we cannot infer either the truth or the untruth of the particular. The correctness of the first inference is obvious at once, and it only requires the removal of a misunderstanding to make the impossibility of the second equally so. A person who denies the universal proposition, 'all S are P,' is usually led to do so by

having already observed some S which are not P; but he will not have included all S in this observation. His intention therefore generally is merely to deny the universal application of the proposition to all S, while leaving its truth in single cases of S undisputed; and thus it is that in ordinary speech expressions such as 'It is not true that all S are also P,' are actually understood to admit incidentally the truth of the particular proposition, 'some S are P.' Logic, on the other hand, knows nothing of these unexpressed suggestions in the denial of the universal proposition: it recognises merely what lies in the expressed negation itself. But it is just this which is ambiguous. For the asserted untruth of the proposition, 'all S are P,' is equally a fact, whether the proposition is true of only some S or of none. So long therefore as this ambiguity is not removed by accessory statements, we cannot infer from the negation of the universal proposition either the truth or the untruth of the particular.

76. Going in the opposite direction, from the particular to the universal or *ad subalternantem*, we infer the untruth of the universal judgment from that of the particular, but not the truth. Here, too, the first conclusion is obvious, if we avoid the ambiguity already alluded to. A person who denies the proposition, 'some S are P,' may, it is true, intend merely to deny that P is confined to some S, and the effect of his meaning that '*not only* some S are P' would then be to affirm the universal proposition 'all S are P.' But just because this consequence would directly imply that the particular judgment, 'some S are P,' also remained true, logic cannot possibly interpret the denial of that judgment in this way. For logic this denial means nothing but that 'there is no such thing as some S which are P'; and what is *not even* true in some cases is still less true in all. Consequently the negation of the particular always negates the universal too. The impossibility of the second inference explains itself; the truth of P in the case

of some S can never prove its truth in all S: it is only because this unjustifiable generalisation of single observations is the commonest of logical mistakes, to which science and culture owe most of their errors, that it is worth while to prohibit with especial emphasis this false inference *ad subalternantem*.

77. Universal judgments are contradictorily opposed to particulars of unlike name, A to O and E to I and *vice versa;* we infer *ad contradictoriam* both the untruth of the one from the truth of the other and the truth of the one from the untruth of the other. The first inference needs no explanation, the second a brief one. If we deny the proposition A, 'all S are P,' the denial is consistent with both the assumptions E and O, 'no S is P,' and 'some S are not P'; but the second, which is included in the first, is true in any case; consequently the truth of O follows certainly from the untruth of A. If we further deny O, 'some S are not P,' this means, according to what we said above, 'there is no such thing as some S which are not P,' and this is equivalent to A, 'all S are P.' If we deny E, 'no S is P,' either all or some S, in any case the latter, are P, and consequently I is true, 'some S are P': if we deny I, this means, 'there is no such thing as some S which are P,' and is equivalent to the affirmation of E, 'no S is P.'

78. The two universal judgments of unlike names are only *contrariwise* opposed, and we infer the untruth of the one from the truth of the other, but not the truth of the one from the untruth of the other. The first case is obvious: the impossibility of the second follows, after what we said before, from the consideration that, while the negation of a universal judgment allows an inference *ad contradictoriam* to the truth of the particular of unlike name, the truth of the latter does not allow an inference *ad subalternantem* to that of the universal to which it is subordinate. Lastly, the relation between the two par-

ticular judgments *I* and *O* is called *subcontrary* opposition. We infer *ad subcontrariam* the truth of the one from the untruth of the other, but not the untruth of the one from the truth of the other. In fact, the two propositions, 'some *S* are not *P*,' and 'some *S* are *P*,' may both be true together; but if one is denied, the truth of the opposite universal follows *ad contradictoriam*, and from this again follows *ad subalternatam* the affirmation of the particular subordinate to it.

79. I may also mention another logical operation which has a kindred object. All observations, which always admit ultimately of being expressed in the form of a judgment '*S* is *P*,' present us only with that combination of *S* and *P* which actually occurs at the moment of observation: they tell us nothing as to whether *S* and *P* will be separable or not in other cases, whether, in fact, there are *S* which are not *P* or *P* which are not *S*. Now we have a practical interest in this question which is very intelligible: we want to know whether a *P* which has occurred in *S* may be considered as a *mark*, enabling us to determine the nature of the subject in which it occurs: in short, whether everything which has the characteristics of a *P* is also always an *S*. The answers to be expected to this question will accordingly take the form, '*P* is *S*'; and they are therefore called *conversions* of the original judgments which gave rise to them. It is also obvious that we have a special interest in knowing whether *P* points to a subject *S* necessarily and always, or only possibly and sometimes; whether, as it is ordinarily put, all *P*, or only some, are *S*. Hence it is the quantity of the original and the converted judgment to which particular attention is paid, and the conversion is called pure (*conversio pura*) when the quantity of the second is that of the first without any change, and impure (*conversio impura*) when it is different, especially when the universal truth of the original judgment has to be reduced to particular,

in order to make it true when converted. The results are as follows.

80. The universal affirmative judgment, 'all *S* are *P*,' understands by *P* either a higher genus in which *S* is contained along with other species, or a universal mark in which *S* partakes along with other subjects. In both cases there is a part of *P* left which has nothing to do with *S*, and only impure conversion can take place into the particular judgment 'some *P* are *S*.' This rule deserves attention, for it is one of the commonest mistakes of carelessness and one of the most favorite means of deception to substitute the universal for the particular inference, and to assert, 'If *P* belongs to all *S*, then *S* belongs to all *P*.' It is true that we do meet with universal affirmative judgments which admit of this pure conversion, those viz. in which the extents of *S* and *P* exactly cover each other, and *P* therefore belongs not only to *all S*, but *only* to all *S*, so that all *P* are also *S*. Such so-called *reciprocal* judgments are, 'all men are naturally capable of language,' 'all equilateral triangles are equiangular'; they can be converted into, 'all that is naturally capable of language is man,' 'every equiangular triangle is an equilateral one.' But it is only knowledge of the matter of fact contained in the judgment in question which can assure us that the relation, upon which this possibility depends, holds good between *S* and *P* in any particular instance. Mathematics, therefore, where the pure conversion of universal affirmative judgments is frequent, are right in demanding special proof in every case of the truth of the converted judgment, and by this caution inculcate the rule that by right of mere logic the universal affirmative judgment admits only impure conversion into a particular affirmative. It is otherwise with the universal negative judgment, 'no *S* is *P*.' This complete exclusion of the two concepts from each other clearly holds good reciprocally, and justifies the assertion that 'no *P* is *S*.'

The universal negative judgment is therefore convertible into another universal negative.

81. The particular affirmative proposition, 'some *S* are *P*,' obviously yields pure conversion into another particular affirmative, 'some *P* are *S*.' And this inference is satisfactory in all cases in which *P* is a universal predicate in which *S* partakes along with other subjects; thus the assertion, 'some dogs bite,' is rightly converted into 'some things that bite are dogs.' But when *S* is the genus of which *P* is a species, as in the proposition, 'some dogs are pugs,' the only logically admissible conversion, 'some pugs are dogs,' will contrast unfavourably with the actually true one, 'all pugs are dogs.' The former is no doubt true also, but it expresses only a part of the truth, and in a form which appears rather to deny than to affirm the other part, that all other pugs also are dogs. We feel this still more if we start with the judgment, 'all pugs are dogs,' and convert it twice over. From the first conversion, 'some dogs are pugs,' we cannot get back again by the second to the original proposition; and thus the logical operations have here resulted in eliminating a part of the truth. This inconvenience could easily be avoided if the expressions of quantity were regarded, as the sense requires that they should be, as inseparable from their substantives; we should then formulate the proposition, in the first instance as follows, 'all pugs are some dogs'; then by conversion, 'some dogs are all pugs,' and by a second conversion, 'all pugs are some dogs.' But it is not worth the trouble to improve what are after all barren formulae.

The particular negative judgment, 'some *S* are not *P*,' as such asserts merely the separability of *S* from *P*, not that of *P* from *S* also. The pure conversion therefore, 'some *P* are not *S*,' does not hold good universally, but only of those *P* which are predicates common to different subjects, and are not therefore exclusively dependent upon the nature of *S* for their occurrence. For this reason the proposition,

'some men are not black,' can be converted into, 'something black is not man'; but the judgments, 'some men are not pious,' 'some are not Christians,' would yield 'something pious is not man,' 'some Christians are not men,' both inadmissible because piety and Christianity, though not belonging to all men, belong *only* to men. These disadvantages are in general only avoided by joining the negation to the predicate, and then converting the proposition, 'some *S* are non-*P*,' like a particular affirmative into 'some non-*P* are *S*'; e.g. 'something not-black, something not-pious, some non-Christians, are men.'

82. The process necessary in this case has been extended to all judgments under the name of conversion by *contraposition:* in the affirmative judgments the negation of non-*P* takes the place of the affirmation of *P*, in the negative the affirmation of non-*P* takes that of the negation of *P*; the judgments thus changed are then converted according to the ordinary rules. In this way we get the following results; first, for *A*, 'all *S* are *P*,' 'no *S* is non-*P*,' and so 'no non-*P* is *S*'; for *I*, on the other hand, 'some *S* are *P*,' the transformation into, 'some *S* are not non-*P*,' would not, after what has been said above, allow any conversion, and contraposition would therefore be impossible; for *E*, again, 'no *S* is *P*,' we get 'all *S* are non-*P*,' 'some non-*P* are *S*.' To carry out these operations in actual instances would produce unshapely and unnatural forms of expression; the substantial meaning of the four forms of judgment may be given more simply by replacing their quantitative determinations by the equivalent modal ones: even the contraposition of *I*, which in itself is impossible, is thus made available. The conversion of *A* would then mean, 'If the predicate *P* belongs to all individuals of a genus *S*, it is impossible for anything in which this mark is absent to be an *S*': that of *I* would mean, 'If *P* is only known to belong to some species of *S*, it is not necessary, but only possible, that something in which *P* is absent should not be

an *S*': that of *E*, 'If the mark *P* is universally absent from, or contradictory of, the genus *S*, it is not necessary, but only possible, that something which similarly lacks or is contradicted by *P* should be a species of *S*'; and the same inference applies to *O* also, 'If some *S* are not *P*, something which also is not *P* may be an *S* but need not be so.'

CHAPTER III.

The Theory of Inference and the Systematic Forms.

Preliminary remarks upon the Aristotelian doctrine of syllogism.

I HAVE pointed out the unsolved problem which compels us to advance beyond the disjunctive judgment. Before I follow up this thread of connexion systematically, I think it will be advantageous to state the theory of syllogism in the form which it received from Aristotle. I shall not however follow the original exposition of the great Greek philosopher, but the more convenient form which came into vogue later. The writings of Aristotle are preserved, and anyone who takes an interest in the origin of these doctrines may easily enjoy his masterly development of them: but when we are concerned, not with the history of the thing, but with the thing itself, it would be useless affectation to prefer the inconvenient phraseology of the inventor to those improvements in detail which subsequent ages have placed at our disposal.

83. Following Aristotle, we give the name of inference or syllogism to any combination of two judgments for the production of a third and valid judgment which is not merely the sum of the two first. Such production would be impossible if the contents of the antecedent judgments, the two premisses, *propositiones praemissae*, were entirely different; it is only possible if they both contain a common element *M*, the middle concept or *terminus medius*, which

the one relates to S, the other to P. This medium brings the two concepts S and P into connexion, and they can then meet in the conclusion in a judgment of the form 'S is P,' or, more shortly, $S\ P$, from which the middle concept which served to produce it has again disappeared. There is no reason in the nature of the case for making a difference of value between the two premisses $S\ M$ and $P\ M$; but a tradition, which cannot be disregarded without subjecting all established rules to a bewildering change of meaning, has decided that the premiss which contains along with M the predicate P of the coming conclusion shall be called the *major premiss*, and that which contains S, the subject, the *minor*; the conclusion itself is always conceived in the form $S\ P$, not in the reverse form $P\ S$. This being presupposed, the further differences in the position which the three concepts may assume give rise to the following four arrangements, of which the first three represent the three figures of Aristotle, while the fourth forms that of Galen.

$$
(1)\ \frac{\begin{array}{c}MP\\SM\end{array}}{SP} \qquad (2)\ \frac{\begin{array}{c}PM\\SM\end{array}}{SP} \qquad (3)\ \frac{\begin{array}{c}MP\\MS\end{array}}{SP} \qquad (4)\ \frac{\begin{array}{c}PM\\MS\end{array}}{SP}
$$

84. If we now ask whether, and under what conditions, these arrangements of premisses, which are in the first instance merely based upon rules of combination, give ground for a valid inference, we find at once that S and P can only be united in the conclusion if the middle concept remains precisely the same; their union is obviously unjustifiable as soon as the M connected with S in the one premiss is different from the M connected with P in the other. Such a division of M would give four concepts in the premisses, instead of the necessary and sufficient three; the avoidance of this *quaternio terminorum*, and the securing of complete identity in the middle term, is therefore the condition of conclusiveness in all figures alike. To fulfil

this condition it is first of all necessary in all figures to exclude any ambiguity in the meaning of the word which denotes the middle concept; but besides this there are special precautions for the same purpose, which the peculiar structure of the several figures renders necessary, and which we have now to mention.

85. In the *first figure* S is included in M in the minor premiss, M in P in the major, and therefore S in P in the conclusion. The idea upon which this inference is based is evidently that of subsumption; that which is a predicate of the genus is a predicate of every subject of the genus. This is of itself sufficient to show that the major premiss in the first figure must be universal; for it has to express the rule which is to be applied to the subject of the minor. The necessity that the middle term should be identical leads to the same result. For the S of the minor premiss is always a definite kind or a definite case of M; this however is not expressed in the form of the proposition; as far as the form goes S might be merely any kind of M in general; if this indeterminate M is to be the same as that which the major premiss asserts to be P, this can only be secured if the major premiss speaks universally of all M, thus including the indeterminate cases along with the rest. It is true that in that case the M expressed in the major premiss is not identical with the M of the minor, which, as predicate of S, necessarily signifies only a part of the whole extent of M; but this apparent difficulty disappears when we consider that the M of the major premiss which is actually *employed* in producing the conclusion is likewise only a part of that which is expressed, that part, namely, which is *intended* in the minor. Further, as the inference in the conclusion depends upon the subordination of S to M, this subordination must be a fact, in other words, the minor premiss which expresses it must be affirmative; if it were negative, it would simply deny the existence of any ground for the validity of the conclusion.

On the other hand it does not affect the logical connexion of the syllogism, but depends merely upon its particular content, whether the major premiss affirms or denies *P* of *M*, and whether the application furnished by the minor of the general rule to an instance embraces all *S* or only some. The quality of the major premiss and the quantity of the minor are therefore unlimited. Lastly, the relation, whether affirmative or negative, in which the major premiss places *M* to *P*, must be transferred unaltered to the unaltered subject, whether universal or particular, of the minor; the conclusion therefore has the quality of the major and the quantity of the minor. If we suppose all the possibilities exhausted for which these rules leave room, we get four valid kinds or moods of the first figure. Their scholastic names *Barbara*, *Celarent*, *Darii*, and *Ferio*, which by the three vowels in order denote (as every one knows) the quantity and quality of the premisses and the conclusion, show at a glance the distinctive feature of the first figure, namely, its capacity to produce conclusions of every kind.

86. The premisses of the *second figure* show us two subjects *S* and *P* in relation to the predicate *M*. If both subjects either have or have not this predicate, i.e. if both premisses are affirmative or both negative, no inference can be drawn from them as to a mutual relation between *S* and *P*. For innumerable subjects may all participate in, or all be excluded from, a mark *M*, without necessarily having any other point in common, and in particular without the one, *S*, being necessarily a species of the other, *P*. Only if the one subject has or has not the mark *M* always or universally, while the other is related to *M* in the opposite way, is there ground for concluding that the second cannot be a species of the first. The premisses in the second figure must therefore be of opposite qualities, and one of them must be universal. As however it is the tradition that this second subject should be supplied by the minor premiss, the premiss in which the first is mentioned, i.e. the major, must be

the universal one. Thus the conditions of the second figure may be summed up as follows: the major premiss is universal, but is not limited as to quality; the minor is of the opposite quality to the major and is not limited as to quantity; the conclusion is always negative, and has the quantity of the minor. The possible moods are *Camestres*, *Baroco*, *Cesare*, *Festino*.

87. The *third figure* brings the same subject *M* into relation to two predicates, *P* and *S*. If *M* has both predicates, i.e. if both premisses are affirmative, the union of *P* and *S* must be *possible*, and the conclusion therefore, according to the usual logical expression of such a possibility, is, 'some *S* are *P*.' The necessary identity of *M* is in this case sufficiently secured by the universality of one premiss, it does not matter which; for it clearly makes no difference whether all *M* have the mark *P* and only some have *S*, or whether all *M* have *S* and only some *P*; in either case there are always some *M* which have both and thereby justify the conclusion, which is always particular, 'some *S* are *P*.' Moreover this case, in which *M* is subject in both premisses, is just one in which its identity might be easily guaranteed by a word of completely individual meaning, the proper name of a person for instance. We often meet with such inferences: in order to prove the compatibility of two actions which seem to be mutually exclusive, we bring forward an instance, e.g. 'Socrates was *P*, and Socrates was also *S*,' consequently 'what is *S* may also be *P*,' or 'some *S* is *P*.' Logic justifies such inferences by attributing to the singular judgment, i.e. one whose subject is not an indefinite part of a universal concept but a perfectly definite and unique individual, the syllogistic value of a universal judgment. Thus this case comes under the above rule, which, where both premisses are affirmative, requires one to be universal, prescribes a particular affirmative conclusion, and admits the moods *Darapti*, *Datisi*, and *Disamis*.

88. Again, if the same subject possesses one of the marks

but not the other, i.e. if one premiss is affirmative, the other negative, *S* and *P* must be separable, or, according to the ordinary phraseology, the particular negative conclusion follows, 'some *S* are not *P*.' In this case also it is sufficient for the identity of *M* that one premiss, it does not matter which, should be universal, but the minor premiss must be affirmative. For though one of two marks which occurs in a given subject is no doubt always separable from the other which does *not* occur in that subject, the latter is not necessarily separable from the former; it is further conceivable that if it exist at all it can only do so in conjunction with the other. Thus life without intelligence is a possible mark of an animal, but not intelligence without life. It is therefore the affirmed mark only which is separable; only of it as subject can the conclusion assert that it is not always combined with the other as predicate; and as this subject of the conclusion is customarily furnished by the minor premiss, the minor premiss must be affirmative and only the major can be negative. Under this condition mixed premisses yield the moods *Felapton*, *Ferison*, and *Bocardo*, these like the preceding ones having only particular conclusions.

89. Lastly, it is asserted by logic as a universal principle that in the third as in the other figures two negative premisses yield no valid inference. This is incorrect; a conclusion may be drawn from them similar in kind and equal in value to those which are derived from affirmative or mixed premisses. For if the first of these prove that *S* and *P* may exist together, and the second that they may exist apart, two negative premisses prove with equal ground that *S* and *P* are not contradictorily opposed, and that accordingly what is not *S* need not therefore be *P*; in ordinary phraseology, 'some not-*S* are not *P*.' I cannot see why this conclusion should stand lower in value than the two others; the first only says to us, 'when you find *S*, be prepared for the possibility of finding *P*,' the second, 'when

you meet with *S* do not reckon upon the existence of *P*,' and similarly the third, 'where you do not observe *S*, beware of inferring for that very reason the presence of *P*.' In life we often meet with such inferences; over and over again, when the necessary presence of some quality has been over-hastily concluded from the absence of some other, we appeal to instances in which neither the one nor the other is found, and so correct an erroneous prejudice by an inference in the third figure from two negative premisses. This conclusion therefore is undoubtedly valid, but it would be an anachronism to invent supplementary names for its various moods.

90. The premisses of the fourth figure, ascribed to Claudius Galenus, are in form the counterpart of the first figure of Aristotle, but do not equal it in value. Its moods are *Bamalip*, *Calemes*, *Dimatis*, *Fesapo*, *Fresiso*. As to the premisses of *Bamalip*, e.g. 'All roses are plants,' 'All plants need air,' every one who thinks naturally will tacitly transpose them, and draw the conclusion of *Barbara* in the first figure, 'All roses need air.' It is true that this conclusion is then of the form *PS*, but the form *SP*, which is required by the fourth figure, can be easily obtained from it by conversion, 'some things that need air are roses.' On the other hand we cannot by conversion recover from this conclusion in the fourth figure the one which we drew from the same premisses in the first; its conversion only yields the particular proposition, 'some things which are roses need air.' Thus in this case the conclusion in the figure of Galen actually loses a part of the truth which is established by the premisses, a bad recommendation for a process of inference, the function of which is always to conclude from what is given as much new truth as possible. This awkwardness could indeed be avoided, as was shown before, but the inference would not thereby be made more natural. Equally unnatural are *Calemes* and *Dimatis*, the premisses of which will always be transposed by the un-

sophisticated mind and applied in *Celarent* and *Darii* of the first figure: they do not indeed occasion a loss of truth, since the negative conclusion of *Calemes* admits pure conversion, while that of *Darii* is particular like that of *Dimatis*. It is only *Fesapo* and *Fresiso* which are less readily reducible to the first figure, owing to the negative minor premiss which results in both and the particular major which results in the latter; by pure conversion of their majors they can be transposed into *Felapton* and *Ferison* of the third figure instead, and this change will have the same effect of making the conclusions more natural. In all points, therefore, the fourth figure is a very superfluous addition to the three figures of Aristotle.

91. Aristotle considered the inferences in all the three figures to be valid, but only that in the first to be perfect, because in this figure only does the ground upon which all inference depends for its possibility, the subordination of the particular to the universal, find formal expression in the structure of the premisses. In the other figures too, indeed, (as he held), the inference rests upon the same principle, and the relations of subordination, which are necessary and sufficient for drawing a conclusion according to that principle, are contained in the premisses and do not need supplementing by information from without; but they are not exhibited in the actual structure of the premisses; we have to look for them there. To make good this formal defect in the two latter figures, Aristotle has shown us how, without any change of content, their premisses may be transformed into those of the first figure. To some people this has seemed superfluous, and they have objected that the two other figures also conclude according to principles of their own and requiring no other evidence: thus the fundamental idea of the second, that if two things stand in contrary relations to the same mark the one cannot be a species of the other, is clear in itself and independent of the principle of subordination. I doubt this, but shall

not pursue the point further; for to hold that the conclusions of the two latter figures are drawn upon any *principle* at all, is to admit that the ground of *all* inferences is the subordination of the particular to the universal; for to what did those figures apply their principles if not to justify the conclusion by subordinating the content of the premisses to them? Aristotle was therefore right in his general idea of the superiority of the first figure; we may also share the interest which he took in justifying the other figures by these changes of form; but it is true that in practice it is seldom of much use to carry them out; in considering the fourth figure just now we seemed to find such a case; the inferences of the second and third figures are too transparent to need this assistance.

92. It is therefore sufficient to mention that in the names of the moods of the two last figures the scholastic logic has indicated the operations necessary for this purpose by the letters *m s p c.* Thus *m* implies the transposition (*metathesis*) of the premisses: *s* and *p* tell us to convert, purely (*simpliciter*) or impurely (*per accidens*), the proposition whose characteristic vowel they follow: the meaning of *c*, reduction to impossibility (*per impossibile ductio*), is the only one which is not quite so simple, and may be at once illustrated by the case of *Baroco.* The premisses here are, 'all *P* are *M*,' 'some *S* are not *M*,' and the conclusion, 'some *S* are not *P*.' If we suppose this conclusion to be false, it follows *ad contradictoriam* that 'all *S* are *P*.' If this were so, and if this new minor premiss, 'all *S* are *P*,' were subordinated to the given major, 'all *P* are *M*,' it would follow in *Barbara* of the first figure that 'all *S* are *M*.' But this result contradicts the given minor 'some *S* are not *M*'; it was therefore wrong to deny the truth of the conclusion in *Baroco*, and that conclusion, 'some *S* are not *P*,' is right. The other operations scarcely need illustrating. We have lately seen how, by transposition, *m*, of the premisses, and impure conversion, *p*, of the conclusion, which was then

drawn in the first figure, *Bamalip* of the fourth is reduced to the first. *Camestres* of the second, 'all P are M,' 'no S is M,' 'no S is P,' gets by transposition, *m*, of the premisses and pure conversion, *s*, of the minor, the new premisses 'no M is S,' 'all P are M,' from which it follows in *Celarent* of the first figure, 'no P is S'; this conclusion further requires pure conversion, *s*, in order to yield 'no S is P,' as required by *Camestres*. *Darapti* of the third figure runs, 'all M are P,' 'all M are S,' 'some S are P'; the impure conversion, *p*, of the minor gives the premisses 'all M are P,' 'some S are M,' and the resulting conclusion in *Darii* of the first figure, 'some S are P,' requires no further transformation, being immediately identical with that of *Darapti*.

93. Thus far we have conceived of the premisses as categorical judgments of the form 'S is P.' But the course of our thoughts may also suggest them in an hypothetical or disjunctive form. These differences, important as they are for the judgments as such, are not so for the formal connexion of the syllogism; they always belong to its content, and it is only necessary to take note of them, not to alter the ordinary syllogistic rules on their account. This is most obvious where we have two hypothetical premisses, in each of which two of the three *propositions* M S P are connected as *protasis* and *apodosis*. Just as with categorical premisses where M S P denote three *concepts*, the inference in *Darii* is as follows: 'P is always true if M is true, M is sometimes true if S is true, therefore P is sometimes true if S is true'; in *Camestres*, 'M is always true if P is true, M is never true if S is true, therefore P is never true if S is true'; in *Disamis*, 'M is sometimes true if P is true, M is always true if S is true, therefore P is sometimes true if S is true.'

The cases are more peculiar when the major premiss is hypothetical and connects universally a consequence F, expressed in the apodosis, with a condition G, contained in the protasis, while the minor is categorical and affirms or

denies either G or F of all or some instances of S. The simplest way is to class these cases with the immediate inferences from judgments, for condition and consequence are related as *subalternans* to *subalternata*. Firstly, then, the fact that the condition G is not true in certain cases of S does not justify us in inferring *ad subalternatam* that the consequence F is not true in the same cases, for the same consequence may arise from other and equivalent conditions. But if the condition is true, we infer the truth of the consequence. This gives rise to two syllogisms, since G may imply either that F is true or that it is not true; (1) 'If G is true F is always true, G is true in all or some cases of S, therefore F is true in all or some cases of S'; this is a *modus ponendo ponens*, which posits the consequence by positing the condition, and it evidently answers to the moods *Barbara* and *Darii* in the first figure: (2) 'If G is true F is never true, G is true in all or some cases of S, therefore F is not true in all or some cases of S'; a *modus ponendo tollens*, in so far as it does away with the consequence F by positing the condition of its opposite, and obviously a counterpart of *Celarent* and *Ferio* in the first figure.

In the opposite direction, *ad subalternantem*, the truth of the proposition F in certain cases of S does not prove the truth of the particular condition G on which it was found to depend in other cases, for the same consequence F may arise from several equivalent conditions. But the fact that F is not true in certain cases of S does prove that all conditions upon which it could depend, and therefore the particular condition G, are not true. The following syllogisms are therefore admissible: (3) 'If G is true F is always true, F is not true in all or some cases of S, therefore in all or some cases of S G is not true,' a *modus tollendo tollens*, which by doing away with the consequence does away with the condition which, had it been true, would inevitably have given rise to it; it corresponds clearly to *Camestres* and *Baroco* of the second figure: (4) 'If G is true F is never

true, F is true in all or some cases of S, therefore in all or some cases of S G is not true,' a *modus ponendo tollens*, which by positing a consequence denies the condition under which it would have been impossible; it repeats *Cesare* and *Festino* of the second figure. Lastly, we may reflect that the fact that G is not true may also imply that F is or is not true, in which case we get the syllogisms, (5) 'If G is not true F also is not ever true, in all or some cases of S G is not true, therefore in the same cases F is not true,' a *modus tollendo tollens* without any peculiarity, merely translating the *ponendo ponens* into the negative: (6) 'If G is not true F is always true, in all or some cases of S F is not true, therefore in these cases G is true,' a *modus tollendo ponens*, which was wanted to complete the possible combinations of condition and consequence, positive and negative; it posits the truth of a condition by doing away with the consequence which would necessarily follow if it were not true. An easy change in the form of expression shows that these two last cases also belong to the second figure; the latter of them might be put thus, 'If non-G is true F is always true, F is always or sometimes not true, therefore non-G is always or sometimes not true.' As this exhausts everything that can be proved from the relation of subalternation, there are no consequences of this kind which could be classed under the third figure.

94. These syllogistic devices are in my mind of less importance than a circumstance which I never find thoroughly considered in connexion with the present subject, the circumstance that all these inferences refer merely to a relation between the *condition* G and its *consequence* F, not to that of a *cause* G to its *effect* F. It is only in the world of thought that a condition G, if it is once supposed to be true, *always* has the consequence which by a necessity of thought belongs to it; in the real world the cause G, even if it exists and is operative, may always have its effect F frustrated by an opposing force U. It being transferred to

actual events, therefore, all these inferences require to be modified in ways which applied logic will show us: thus it is not allowable to conclude that wherever the cause G operates its effect F is necessarily a *fact*, nor to assert that, if G is a cause of hindrance to F, where this hindrance exists F cannot exist; G also in its turn may be hindered by a U, or F may be realised in spite of it by a third cause V. In pure logic, therefore, it is quite an improper description of the cases which we have been dealing with to say, that their minor premiss expresses the real existence of G or F; the truth is that these two simple letters stand here for judgments of the form 'S is P'; it is only the logical admissibility or necessity of this connexion of thought between S and P which the minor premiss asserts in regard to certain cases of S, while the major connects it with another similar relation between S and Q, so as to form an hypothetical judgment of universal validity. I will not pursue this point further here; I have made my exposition somewhat prolix in expression with the view of indicating how the matter really stands.

95. If it is true of a subject Z that it is either P, Q, or R, or that it is both P, Q, and R, or that it is neither P, Q, nor R, we first substitute for this triple predicate the simple U, and call U in the first case disjunctive, in the second positive, in the third negative. If anyone takes the not absolutely necessary trouble to follow the application of such disjunctive, copulative, and remotive premisses in the syllogism, he will find these results. (1) If the major premiss is $Z\ U$, and in the minor $S\ Z$ an S is subordinated to Z, the ordinary conclusions $S\ U$ of the first figure follow, and U has in them the same meaning as in the major: (2) If the universal major is $Z\ U$, the minor $S\ U$, and U is in one of them positive or disjunctive, in the other negative, we get the negative conclusions $S\ Z$ of the second figure with the quantity of the minor: (3) from the major $U\ Z$ with a positive or negative U, and the minor $U\ S$ with a U of the

same or the opposite quality, there result the conclusions *S Z*, always particular, of the third figure: (4) in the two latter cases, where *U* having become the middle term disappears from the conclusion, its multiplicity is entirely without significance; what follows follows all the same if the position of one only of its members in the two premisses be taken into account. The result is equally little affected if the universal major *Z U* has a minor which affirms or denies of the particular subject *Z* one of the members of *U*. If the major distinguishes only two alternatives and says, 'all *Z* are either *P* or *Q*,' and the minor 'this *Z* is *P*' or 'this *Z* is not *P*,' it follows that 'this *Z* is not *Q*' or 'this *Z* is *Q*.' These consequences explain themselves from the nature of contradictory opposition; they can be reduced, but without any conceivable advantage, to the first figure; 'every *Z* which is not *P* is *Q*, this *Z* is a *Z* which is not *P*, therefore this *Z* is a *Q*.' The same unfruitful reflexions may be extended to a *U* of more than one member in the major premiss, for we can always make any number that we choose of its members into the subject, and say (with only a bipartite *U*), 'every *Z* which is not *P* and is not *Q* is either *R* or *T*.' Lastly, *polylemmas* (dilemmas, trilemmas) are syllogisms with a disjunctive *U* of many members in the major *Z U*, and the same number of minors, which taken together affirm of each one of the members of *U* the same further consequence *T*. These are not cases of new logical forms but only new applications of old ones, and we may return to them in our applied logic.

96. On the other hand, I have no intention whatever of coming back to the doctrine of *chains of inference*. Every conclusion of a syllogism may conceivably become the major premiss of another syllogism: the first is then called the *prosyllogism* of the second, and each one that follows the *episyllogism* of the one which preceded it. A mere comparison of the names of the moods shows us at once many properties of the chain thus produced. If its last

member is to be universal, the whole series of prosyllogisms, and therefore the whole chain, must be in the first two figures; the entrance of any member in the third figure produces a particular conclusion, which never leads back again to universal conclusions. If one of the syllogisms has a negative conclusion, the conclusions of all episyllogisms are negative; and a chain can only end with a conclusion at once positive and universal if it is in *Barbara* through its whole course. It is moreover usual to require, on the analogy of the simple syllogism, that the major premiss of the first prosyllogism should furnish the predicate *P* of the ultimate conclusion, and the minor of the last episyllogism its subject *S*: it would only need patience to find the rules for the formation of such a series, but I cannot see of what use they would be. If the conclusion of a prosyllogism, which is also the major[1] premiss of the episyllogism, is not expressed, the series give rise to the two forms of *Sorites*. The Aristotelian form, '*A* is *B*, *B* is *C*, *C* is *D*, therefore *A* is *D*,' includes each concept in the one which follows; it thus proceeds from the lower to the higher, and is produced by suppressing the conclusions, which we could elicit from each pair of members as follows, '*B* is *C*, *A* is *B*, therefore *A* is *C*,' and then, '*C* is *D*, *A* is *C*, therefore *A* is *D*.' The other form, a late discovery of Professor Goklenius of Marburg (1547–1628) takes the opposite direction; its premisses, '*B* is *A*, *C* is *B*, *D* is *C* . . .,' suppress the conclusion of the two first members, '*C* is *A*,' which as major premiss to the third gives the conclusion of the chain in the first figure, '*D* is *A*.'

A. *Syllogistic Inference. Inference by Subsumption. Inference by Induction. Inference by Analogy.*

97. The logical truths of which the mind had gradually become conscious in dealing with its ideas were provision-

[1] ['*Minor*' premiss, in the Aristotelian *Sorites*. The author's words only apply to the Goklenian form.]

ally summed up by the disjunctive judgment as follows: every S, which is a specific form of M, possesses as its predicate a particular modification of each of the universal predicates of M to the exclusion of the rest. The problem which remained was to discover the intellectual operations by which this required specific mark could be determined for a given S. This problem is not solved by the Aristotelian syllogisms; they confine themselves to placing the subject of their conclusion in relation merely with the universal form of the predicate mentioned in the major premiss; so that in spite of the manifold development given to them and their possible varieties by the acuteness of earlier logicians, they are merely the expression, formally expanded and completed, of the logical truth already embodied in the disjunctive judgment. Like the impersonal judgment, which, by distinguishing subject and predicate, made formally explicit a division already indicated in the concept, without telling us anything new about the mutual relation of the members thus produced, the Aristotelian syllogism in its first and most perfect figure, to which we mentally refer the others, merely distinguishes in two separate premisses the universal rule and its particular application, which were already similarly related in the disjunctive judgment. Thus the Aristotelian syllogisms, constructed as they all are on the principle of placing one concept within the circuit of another without further defining its position, may be included, under the general name of *inference by subsumption*, and considered as the first and most elementary form of the new group of intellectual operations. We will now attempt to show what is the next step in advance which they compel us to take.

98. As the most graphic illustration of the idea upon which inference by subsumption is based I choose the mood *Darii*[1], which expressly brings a particular case in the minor premiss under the universal law contained in the

[1] [*Sic.* According to ordinary rules the example is in *Barbara.*]

major. 'All men are mortal,' says this mood, 'and Caius is a man,' whence it concludes, 'Therefore Caius is mortal,' clearly meaning that by this conclusion a truth which was not established before is now made certain by the truth of the two premisses and their relation to one another. But as early as the scepticism of antiquity the objection was made, that it is not the premisses which guarantee the truth of the conclusion, but that the conclusion must already hold good in order that the premisses may do so. Where, indeed, would be the truth of the major premiss, 'all men are mortal,' if it were not already certain that Caius participates in this property? And where would be the truth of the minor premiss, 'Caius is a man,' if it were still doubtful whether among the other properties of humanity he had that of mortality also, which the major itself alleges as a universal mark of every man? Instead then of proving the truth of the conclusion by their own independent truth, the two premisses themselves are only true on the supposition of its truth, and this double circle seems at first to make the syllogism logically quite inoperative.

99. The weight of this objection is not to be got rid of by denying it: we will follow out its applications in various cases. If we suppose the major premiss MP to be an analytical judgment, if, that is, we assume P to be a fixed mark without which the content of M cannot be completely conceived, then certainly the universal validity of the major is independently established; but then the minor cannot subordinate an S to M without already attributing to it this indispensable P, that is, without presupposing the conclusion in which that attribution ought first to find expression. If for instance we reckon weight in the concept of body, we form the major premiss, 'all bodies have weight,' without fear of contradiction; but we cannot go on in the minor to call air a body without involving the thought that air too is heavy, which we are not supposed to know until the conclusion. In general terms, the principle of subsumption

requires that the subordinated individual should share the marks of its universal; but, conversely, nothing can be subordinated to a universal without already having the marks which the universal prescribes to it.

The case would be different if we supposed the major premiss $M\ P$ to be a universal synthetical judgment. Then the content of M could be fully conceived without involving the conception of P, though at the same time we should be certain, on whatever grounds, that P is always combined with M. The minor premiss would then merely have to show in S the marks which make it an M, and then, and not till then, the conclusion would add the P which belongs to S in virtue of its subordination to M, but which had not before been part of the conception. In the practical employment of subsumptive syllogisms these assumptions are always made. When we assert, 'all men are mortal,' we conceive the physiological character of man to be fully determined by the rest of his known organisation, and regard mortality as a mark which need not be explicitly thought of when we mentally characterise him, because it follows inevitably from the organisation which determines our conception. And thus in the case of Caius it is enough to establish in the minor premiss the fact that he has this essential organisation, in order in the conclusion to ascribe to him its inevitable consequence. This is still more clear if we conceive the major premiss as hypothetical, and think of P as not a fixed and permanent but a fluctuating mark of M, a consequence which follows upon M under a certain condition x, a mark which under this condition M assumes or loses, a state into which it falls, or an effect which it produces. Then we have merely to subordinate S to M in the minor premiss in order to conclude that S also, if the same condition x operates, must exhibit the mark P. And as a matter of fact this is the form to which most of the effectual applications of the syllogism in science are reducible; they almost all show that S, being a species of M,

will develop or experience under the condition x the same general effect P as we know in M. But as before with the analytical major premiss the question arose, with what right the minor could be asserted, so here with a synthetical major the question arises, with what right we can affirm the universal validity of this major itself. Mortality is to be a new mark, necessarily accruing to the organisation of man: but this universality can only subsist on the assumption that the conclusion is true, and it falls to the ground if some capricious Caius is found who does not die. It is clear what the answer to this will be: 'of course,' it will be said, 'every universal major premiss is false if there is a single instance in which it is not confirmed, and there is always this danger when the universal in question has been formed only by an unjustifiable generalisation from a number of observed instances: but where the necessary connexion of M and P is inherently demonstrable, the very universality of its truth provides against the contingency of a single capricious instance which might contradict it. In the example before us the matter is doubtful: to the ordinary mind the universal mortality of man is only an assumption based upon the overwhelming impression of countless instances, to which as yet no contradictory instance has been found: to the physiologist, as a consequence of the known human organisation, it is certainly a matter of settled conviction, but not one which can be proved with the exactness he would wish. But in other cases the universal validity of the synthetical major premiss is guaranteed either by an immediate perception, or by proofs which reduce a given matter to such a perception, and in these cases the syllogism suffices for securing a particular piece of new knowledge; for all that this requires is perfectly practicable, viz. the subordination of an S to an M, which here really fulfils the function of a middle term in connecting S with a previously unconnected P.'

100. I leave it for the present an open question whether,

and how far, the immediate perception of the universal truth of a synthetical judgment is possible; for so much is at once clear, that in any case we shall be only very rarely in a position to rest the content of a universal major premiss upon this ground; countless universal judgments are expressed and used for inferences, without the possibility of either themselves passing for immediate perceptions or being reduced to such by any practicable method of proof. This wide field of intellectual activity cannot be simply set aside as invalid, nor can it subsist without logical rules of its validity. These rules we have to look for, and there are two which we want. For the effective use of the syllogism it is, firstly, necessary that we should learn to find universal major premisses, based neither on an immediate certitude nor upon the antecedent experience of their truth in every single instance; it must be possible to assert the universal mortality of men, both before it is understood as the necessary consequence of certain conditions, and also before we have tested every individual man to see whether he is mortal. A second rule is necessitated by the minor premiss. There are many cases in which we are able to subordinate an *S* to *M* because we have found in *S* *all* the marks which *M* prescribes to its several species, but in most cases this is impracticable; even in the case of the Caius of our minor premiss no one will consider it necessary or possible, that in order to acquire the right to put him in the genus man we should test all the properties of his organisation. If then the really fruitful exercise of thought is to be possible, there must be a method for finding minor premisses which subordinate a given subject to a genus before it has been shown to possess fully all the marks of that genus. The two methods which I am here requiring admit (though this is not of essential importance), of being attached to somewhat modified forms of the second and third Aristotelian figures.

101. The problem of all inferential processes is naturally

this, from given data or premisses to develop as much new truth as possible; how this is done, is in itself quite immaterial; the method will be determined by the form of the premisses, and these we have to take as experience, internal or external, offers them. Now it often happens that the same predicate occurs or does not occur, not only in two, but in very many different subjects P, S, T, V, W, and the question is, what consequence can be drawn from the premisses, PM, SM, TM, VM, . . . , which belong in form to the second figure of Aristotle. It is clear that in their multiplicity they do not suggest an inference which would connect together any particular two of their subjects; so far as we aim at such an inference, we can only effect it by confining ourselves with Aristotle to two premisses and observing the rules of the second figure. But it is equally open to us to try whether this recurrence of M in such different subjects tells us anything about the significance of M itself, which accordingly would not disappear in the conclusion. Such an experiment is what the natural mind infallibly makes when experience furnishes such premisses, and it is guided in its experiment by the universal principle which dominates all its activity, the principle of translating a given coexistence of ideas into a coherence between their contents. Where we observe the same mark in different subjects, we are predisposed to think that the agreement is not a chance one, and that the different subjects have not therefore stumbled upon the same predicate each through a special circumstance of its own, but are all radically of one common essence, of which their possession of the same mark is the consequence. P, S, T, V will accordingly be different, but still co-ordinate as species under a higher concept Σ; it is not as different individuals, but only as species of the genus Σ, that they bear the common mark M as a necessary mark of that genus. Our conclusion therefore runs as follows, 'all Σ are M'; and in this conclusion Σ stands for the higher universal to which we subordinate

the individual subjects, and for the true subject of the *M* which before appeared as a common attribute of those individuals. Such a process of inference is the simplest case of *Induction*, and under this name forms our *second* member in the group of inferences based upon the subordination of manifold elements to the unity of a universal.

102. This process however seems only to solve imperfectly the problem which was set to it, that of producing universal major premisses for subsumptive syllogisms. For everybody agrees in objecting to induction, that if it is complete its information is certain but not new, while, so long as it is incomplete, it is new but not certain. If *P*, *S*, *T*, *V* are all the species of Σ which exist, and if each already has a premiss informing us that it is *M*, the conclusion can only sum up these premisses in a universal judgment, 'All Σ are *M*'; but it cannot even logically be changed into the general judgment, 'Every Σ as such is *M*'; on the contrary, it remains quite uncertain whether the species of Σ merely participate as a fact in the common *M*, and each ultimately for a special reason of its own, or whether the universal nature of Σ really contains the one and selfsame reason which makes *M* a necessity to all its species. If, on the other hand, besides those subjects which are combined with *M* in the premisses, there are other species of Σ of which those premisses say nothing, then the conclusion is an unjustified inference *ad subalternantem* from the truth of a limited number of instances to the truth of all, an inference which may have probability in various degrees, but never reaches certainty.

It appears to me, however, that these observations, right as they are in themselves, confuse the pure meaning of a logical form with the difficulties of its effective application, and that there was the same error in the criticism made upon the value of the Aristotelian syllogism. The leading idea of that syllogism, that every individual derives its right

and obligation to the possession of its predicates through dependence upon its universal, is without doubt logically a perfectly valid principle, and exhibits in its true light the internal construction of the content of thought in question. It does not lose this logical significance because the truth of the universal includes or, if we prefer it, presupposes its truth in all particular instances; on the contrary, the very meaning of the syllogistic principle is that the two are inseparable. Whatever therefore may be the way by which in practice the mind has *arrived at* the truth of the premisses, when they are once *found* the first Aristotelian figure does express by its structure the inner connexion of the completed content of thought, though it probably does not at all express the division of intellectual labour by which we made it our own. Considered in this way the subsumptive syllogism is the logical *ideal*, to the form of which we ought to bring our knowledge, but it is not the general instrumental *method* by which we compose that knowledge out of the material given to us.

I have a similar remark to make about *induction;* the logical idea upon which it rests is by no means merely probable, but certain and irrefragable. It consists in the conviction, based upon the principle of identity, that every determinate phenomenon M can depend upon only one determinate condition, and accordingly that, where under apparently different circumstances or in different subjects P, S, T, U the same M occurs, there must inevitably be in them some common element Σ, which is the true identical condition of M or the true subject of M. It would be quite unjustifiable to object, that as a matter of experience the same consequence M is often produced by different equivalent conditions, and the same predicate M may occur in extremely different subjects. Such an objection just shows the confusion, which we condemned above, between the logical rule and the conditions of its application. If there are two equivalent conditions for a result M, it is not in

virtue of that which makes them different, P or S, but of that which is the ground of their equivalence, that they are really conditions of the same result: so long as we cannot separate this common characteristic in the two, we have not yet found the true Σ of the conclusion, and have not therefore carried out the induction in the way in which it demands to be carried out. Again, if the same M is found as predicate in a number of extremely different subjects, and subjects (as is usually the case in practice) the several sums of whose marks are only partially known, we may of course make a great mistake if we combine what is common to the known marks of all of them, and then assume it to be Σ, the true subject of the mark in question M. I do not deny that in the practice of induction we are often placed in such unfavourable circumstances; but all these difficulties in carrying out the inductive principle do not alter its universal logical validity when it asserts, that wherever different conditions have the same result M, or different subjects the same predicate M, there must be discoverable one and only one quite determinate Σ, forming the single invariable condition or the single true subject, to which the predicate or the result M is to be universally and necessarily ascribed in a conclusion of the form, 'every Σ is M.' We leave it to applied logic to observe the rules by which we may succeed in discovering this Σ.

103. I introduce the third form of this group under the somewhat arbitrary name of the *inference of analogy*. In the third Aristotelian figure, MP, MS, as in the second, the structure of both premisses being exactly the same, there is nothing in their position to lead us to distinguish major from minor, or to limit their number to two. On the contrary, experience will often show us a larger number of them, MP, MS, MT, MU; will show us, in other words, that a number of different marks does or does not occur in the same subject. These data cannot be rejected by the mind, and it employs them to form an inference which is just like

the one described above, only in the reverse direction. Here, as there, it is guided by the assumption that the different predicates have not united in the same subject *M* by a number of unconnected chances, but that they must be coherent and owe their coexistence to the presence of *one* condition; they belong to *M* because *M* is a Π, and it is this sum of marks which in its completeness constitutes the nature of Π; and *M*, being a species of Π, has a right to unite them all in itself. Thus from these premisses we form the conclusion, 'M is a Π,' and have so executed our second task of finding for the subsumptive syllogism a minor premiss by which a concept *M* (there called *S*) is subordinated to another concept Π (there called *M*).

104. Yet this task, like the former one, seems to be but badly executed, for analogy, like induction, is liable to the charge that, if complete, it tells us nothing new, and if incomplete, nothing certain. If the premisses already give *M* the marks necessary to make it a Π, we gain nothing in knowledge of fact by actually bringing it under this concept; the change is merely in the form of our apprehension of the given content. But in most cases the premisses give only a part of the predicates necessary to Π, and from the presence of these we conclude without certainty to that of the rest, by which alone the whole of Π is realised in *M*. When we have to do with concrete objects, which in their totality consist of countless marks, in great part unknown to us, in part difficult to observe, this is always the case: from a few properties which we actually observe in an object, we conclude that it is a metal, an animal of a certain kind, an instrument for a certain purpose. It is needless to say that numerous mistakes in the employment of analogy arise from this fact; but here also the difficulty of the application does not diminish the value of the logical principle. That principle asserts, that no rightly conceived content of thought consists of an unconnected heap of

marks, which we may increase at pleasure by adding no matter what new elements; what other marks as yet unobserved can combine with the observed marks and what cannot, is already decided, not indeed by *one* mark, but by a given combination of several, in which each is determined by all the rest; this is why we are able from the incipient form of *M* given us by the premisses to infer its further completion and continuance; there is always therefore one and only one Π, which makes legitimate and possible the union of marks given in *M*, and at the same time the addition of others not given. This ideal of thought, which in itself is quite true, only requires, like every form of thought, to be realised in suitable, not unsuitable, matter. It is not any casual pair of predicates in an *M* which suffice for inferring the rest; many such combinations may belong to some other concept Π^1 or Π^2 as well as to Π; in contrast with such unessential marks we shall require essential ones in the premisses, a requirement which is always made in practice, and which it is left to special knowledge of the matter in question to meet. The most important source of inexactness, however, is that all the forms of inference hitherto mentioned give the predicates only a universal form, without indicating their measure, specific modifications, and mutual determination. So long as the premisses only say, '*M* is heavy,' '*M* is yellow,' '*M* is liquefiable,' etc., we certainly find in such data no decisive ground for pronouncing *M* either to be gold or to be sulphur: but this is just why such premisses are only met with in abstract logic; in actual practice attention is always given also to the particular amount, nuance, and combination of the predicates, and from this incipient characterisation its continuity with the completed Π is inferred. It is just this universal practice of the natural mind for which we have to find a theoretical basis in new logical rules, and these we must now go on to consider.

B. Mathematical inferences. Inference by substitution. Inference by proportion. Constitutive equation.

105. I will put together once more, and from different points of view, the motives which impel us to go beyond the syllogisms and look for new forms of thought, and for this purpose I will first touch upon the nature of the judgments which the ordinary theory conceives of as members of the syllogism. In judgments of the form '*S* is *P*,' as I have already observed, language expresses the predicate with a universality with which it does *not* belong to its real subject, and logic usually concedes this when it asserts that not only does the predicate contribute to the determination of the subject, but the subject also to that of the predicate. When we *say*, 'this rose is red,' we do not *mean* that it has a general indefinite red, or any casual shade of colour which happens to be included under the name red; it is rose-red only that we always have in our mind, or, more accurately still, the precise red of *this* rose. If then we wished to express our thought exactly, we should have to say, 'this rose is red with the redness of this rose.' In this apparently quite barren proposition the logical activity would show itself in the fact, that the perceived property of the rose is no longer apprehended as an isolated thing, without any other home in the world; in regarding it as a kind of red in general, which occurs elsewhere and holds good independently of this instance, the mind, as we said before[1], objectifies its perception; it gives to what is perceived a definite place in the world, which makes it something on its own account, and not a merely subjective excitation of the percipient at the moment. In this lies the logical gain which always results when the particular content of a perception is replaced in the judgment by the universal of which it is an instance. But at the same time of course there will be a logical loss, if we get no further than the

[1] [Above, § 3.]

expression of this universal, and if the other part of the perception does not get its due by addition of the particularisation which is necessary to make the universal *named* equivalent to the individual *intended.* This loss is sustained by all ordinary judgments of the form just mentioned, and the Aristotelian syllogisms too confine themselves to dealing with the universal M or the universal P.

106. In this way they leave unsolved the particular problem which the disjunctive judgment suggested, and fail generally to satisfy the practical needs of thought as a living process. For already in the disjunctive judgment it was asserted, that it is not the universal predicate of its genus which belongs to the individual, but a definite modification of it, p, to the exclusion of the rest. What this p is, ought to have been made out by the syllogism; and it could only have done so by supplying to the major premiss, which connects the genus with the universal P, a minor bringing out the peculiarity in virtue of which S is this particular species of the genus and no other, and must therefore have for predicate this and no other modification of P. This has not been done; the minor premiss also only mentioned generally the subordination of the individual to the genus, but not its specific difference from other species of it; hence the conclusion could only say what belongs to the individual as *a* species of its genus, not as *this* species. It hardly needs to be further explained that this falls short of what the actual processes of thinking demand. If we argue, 'heat expands all bodies, iron is a body, therefore heat expands iron,' or, 'all men are mortal, Caius is a man, therefore Caius is mortal,' everyone will feel the barrenness of this procedure, and will reply, 'Undoubtedly heat expands all bodies, but each body in a different degree; undoubtedly all men die, but the liability to die in one man is different from that in another; what we want to know for technical purposes or for administering a life-insurance company is, how iron expands in distinction

from lead, or how the mortality of Caius is to be estimated in distinction from that of other men.' This then is what the new forms have to do; they have to make the individual felt as a definite species of the universal, and so enable us to argue from its distinctive difference to its distinctive predicate.

107. From another point of view we may notice the fact, that in logic it has been too exclusively the custom to use categorical judgments as illustrations, and therefore also to represent the inclusion of one concept within another as the most frequent and most important of logical operations. In the living exercise of thought this is by no means the case; we are seldom concerned in practice to determine a mark which belongs to a concept once for all, or in the circuit of which the concept is to be classed; most frequently we want to know what variable mark P will occur in a concept S if S is subjected to the condition x. Questions of this kind are being raised at every moment by life, science, and art. We must admit that the ordinary syllogistic method does not entirely overlook such cases; but it is only an imperfect way of dealing with them to make P the universal result of the coexistence of x with M in a major premiss, and then to ascribe P to S, again only universally, by subordinating S to M or Mx. What good is it to say, 'if a man is offended he gets angry, Caius is a man, therefore if he is offended he will get angry'? What we want to know is, *how* Caius, being the person he is, will get angry, and consequently how far we may go with him. The subordination of Caius to the concept of humanity helps but little to answer this question; we must look for the special characteristics which distinguish Caius from other persons, and must then have the means of calculating the effect which offence will have upon these characteristics. This may be briefly expressed thus: our inferences cannot be derived from extensive relations between the given concepts, but only from their content; without making the

unprofitable circuit through the universal genus, we have to determine directly from the given marks of a subject, and from the accruing condition x, what new marks will show themselves or what changes will take place in the old ones.

108. Considered from this point of view, the new forms which we have to look for group themselves with the inferences from analogy. For these also concluded from the presence, absence, and combination of certain marks in an S the necessary presence, absence, and mode of attachment of other marks in the same subject. We may doubt indeed whether such inferences from content to content, from mark to mark, are possible on merely logical grounds, and whether the few which really are possible are not already anticipated by the familiar logical doctrines of the compatibility of disparate predicates, the incompatibility of contraries, and the necessary choice between contradictories: statements such as, 'where p is there q must be,' will after all (it may be said) be supplied by experience alone, with the single exception, with which we wish to have no more to do here, when q is already included in the content of p or p in the extent of q. In itself this doubt is right; all assertions about the necessary connexion or incompatibility of two predicates, with the exception of the cases last mentioned, can never be based upon any evidence but that of observation; but it is still a question whether logic, with the means hitherto at its disposal, has made even these necessarily presupposed facts yield all the consequences which they might be made to yield: that it has not done so, I can show more shortly by exhibiting the actual forms of inference to which I refer: in the natural use of the mind they are current and familiar, and all that is done here is to give them the place which belongs to them in a system of logic.

109. Let us leave to the major premiss of our new figure the form, 'all M are P' or $M=P$; to the minor however

we will give, not the indefinite form, 'S is an M in general,' but the definite one $S = sM$; that is, S is that species of M which we get if we conceive the whole structure of the marks in M as determined or modified by the influence of a specific condition s. The conclusion will then have to be, 'S is σP,' and it will assert that S, so far as it is this distinctive species of M characterised by s, possesses, not the universal mark P, but that specific impression of it, σP, which the influence of s must produce in the structure of M. To avoid misunderstanding, it should be observed that the influence of a condition s upon the whole structure of M may transform the different marks of M in extremely different ways; each one of these transformations is a result of s, and on that account I have employed the kindred letter σ in σP: on the other hand it is not generally right, though it may be so in particular cases, to make the modification of a mark equivalent to the modifying condition; therefore the conclusion here could not be indicated by sP. In the form however which we have given to the conclusion, it would be merely the indication, not the solution, of a problem. What is wanted is to give a name to this σP, and to show how P is changed by the influence of s upon M. This remains impracticable so long as we produce M merely in this simple form of a universal concept provided with a name: in order to know how s influences M, we must analyse the content of M into its several parts, and observe in what manner they combine. Nobody, for instance, will undertake to judge how the working of a machine will change under the influence of a force s, so long as he merely has the machine before his eyes as a simple object of perception, M, a steam-engine in general; he must first get to know the inner structure, the connexion of the parts, the position of a possible point of action for the force s, and the reaction of its initial effect upon the parts contiguous to that point. Accordingly, it is only by *sub-*

stituting for the condensed expression or concept M the developed sum of all its constituent parts, with attention to their mutual determinations, that we can hope to follow the influence of s, and so determine, firstly, what is the whole nature of S which $= s\,M$, and, as a consequence, what is the modification $\sigma\,P$ of the predicate P which belongs to this S. As a matter of fact, this second part of the problem is always included in the first; the specific modification of a particular predicate for S cannot possibly be found without first finding the total change produced in M by s, on which the modification depends; for if P were part of a different concept N, the effect on it of the same condition s would not be the same as when it is a part of M. For this reason I shall take no more notice of the inference to $\sigma\,P$, but shall consider the problem of the new form to be to determine $s\,M$, and give it therefore the form,

$$\begin{array}{ll} \text{Major premiss} : & M = a \pm bx \pm cx^2 \ldots \\ \text{Minor premiss} : & S = sM. \\ \text{Conclusion} \quad : & \overline{S = s(a \pm bx \pm cx^2 \ldots)} \end{array}$$

from which, in regard to single predicates, e.g. b, there would follow the definite conclusion, 'S is $s\,.\,bx$,' instead of the indefinite one, 'S is bx.'

110. There is always a danger in expressing very different and yet connected cases by the simplest possible symbols; to avoid misunderstanding, therefore, I add the following observations. By a, b, c, x I wish to be understood, speaking generally, different marks of a concept M, which, when completely enumerated, constitute the whole of M. But in each different concept these marks stand in the most different kinds of relation to one another, and these relations are not expressed in my formula; the double sign $\pm$ has been employed as a faint indication of their possible variety. These signs, $+$ and $-$, do not suffice for a full expression even in a case where M does not mean a

conceptual content of qualitatively different marks, but a mere whole of quantity composed of the commensurable quantitative parts a, b, c, x. The only symbol of a more exhaustive kind would be that of the mathematical function in general, which we used before, $M = F(a, b, c, x\,..)$; but this would have the disadvantage of merely calling up to thought *all* modes of connexion between the parts, without giving a sensuous illustration of any. The form of the series $a+bx+cx^2$ is also an arbitrary symbol; the x only indicates a possible difference of value in the marks, one of which, x, leaves only one other, a, entirely free, while it accompanies the rest as a determining condition. The s of the minor premiss and conclusion appears here as a multiplying factor; this is similarly intended to represent to sense, by the simplest and most familiar form in which one quantity can influence another, the countless different ways in which any concrete condition may act upon the manifold content of any given subject. If we express by a letter placed underneath on the right any kind of change produced by a condition in any kind of given matter, and represent M as a function of a, b, c, x (i.e. $M=\phi(a, b, c, x)$), we should in general only be able to represent the conclusion by $S=\phi_s(a_s, b_s, c_s, x_s)$, not by $S=\phi(a_s, b_s, c_s, x_s)$; for it is obvious that the effect of s may not always be merely to change the single marks, retaining their general connexion (as expressed by the second formula), but also (as expressed by the first) to change this connexion itself; in fact, a condition may so transform the whole structure of a concept that in its new shape it has to be subsumed under a different concept M^1 or N instead of the previous M. The admission which I have now to add makes it unnecessary for me to go further into this point.

111. The advantage which we anticipate from this figure of *syllogism by substitution*, the first of this second group, depends ultimately upon our knowing what the several parts of the conclusion mean, i.e. what that value a_s or bx_s

is which arises from the influence of s upon the developed expression of M. This, however, if it is not to be learnt simply by experience, can only be arrived at by thought if all these mutually related parts are pure quantities, and the relations between them those of mathematical combination and separation. Thus the effect in use of our figure is confined to the region of mathematics, and primarily to the relations of pure quantities. Only the peculiar nature of numbers, each one of which has an expressible relation to every other, allows us to disclose the hidden content of M, by substituting its quantitative parts, in such a way that the condition s can really operate upon it, and that by applying the various rules of calculation, by cancelling incompatible and compounding compatible elements, the change which s necessitates in M can be really carried out and the form of the new result exhibited. On the other hand, if we replace commensurable quantitative parts by incommensurably different marks of a concept, these advantages disappear again; the content of M is only imperfectly disclosed by such a method of substitution; for we have no rule here, as we have in the case of numbers, by which to measure the effect of a condition acting upon these heterogeneous elements. It is true that even in such cases we apply the general idea of substitution: if we want to know how a condition s will act upon a thing, of which we have only the concept M which its natural history supplies, we analyse M into its marks; but the calculation of the effect which s will have on each and all of them, is based merely upon more or less indefinite analogies, suggested by experience or some chance feeling of probability.

112. The fact that the use of the syllogism by substitution is confined to mathematics, cannot hinder us from giving it a place in the systematic series of forms of thought. For in the first place we must not forget that calculation in any case belongs to the logical activities, and that it is only their practical separation in education which has concealed the

full claim of mathematics to a home in the universal realm of logic. But it is not only because they are indispensable to a part of the work of thought, that these forms have their place here; even in those cases where their demands cannot be realised, they are still the ideals of our logical effort. For if they can be applied directly to none but quantitative relations, it is true on the other side that wherever we are quite unable to reduce the object of our investigation to those relations, our knowledge of it remains defective, and that no other logical form can then help us to the answer which a mathematical treatment of the question, if it were practicable, would give us. It is hardly necessary in our days to draw attention to the fact, that natural science owes its existence to mathematics; in other fields also we have learnt to prize the important aid of quantitative statistics in discovering the laws which govern the combinations of society; and even in sciences which from the nature of their objects are farthest removed from mathematics, we often feel very clearly the need of connecting them with quantitative ideas. Moral philosophy may decide that every crime is punishable, without needing a mathematical justification for the assertion; but every punishment which has really to be inflicted must have a measure, and this must be regulated by the measure of badness in the criminal will which has to be punished. If only it were practicable, the penal law itself would draw conclusions in our figure of syllogism; it would break up every crime by substitution into its several elements, and from $s\ M$, i.e. by calculating the particular values of the single elements of the crime in this instance, and so the particular value of the whole, it would deduce $\sigma\ P$, i.e. the kind and amount of punishment which the particular instance deserves.

113. There are other things however besides pure mathematics, and science has certainly succeeded in establishing links of connexion, even between incommensurable phenomena or attributes, which allow us to infer from one to

another. For logic on its part the next problem must be, to look for the forms in which such inference is possible, and so to supplement the imperfection of the substitutive syllogism. It would partly seem indeed that science has only succeeded in thus bridging the incommensurable by doing away with the incommensurability, and showing that two facts, a and b, which at first appear to our perception entirely different in quality, really depend upon quantitative differences between commensurable circumstances: I may recall how physics has reduced the qualitative differences of our sensations of colour, tone, and heat to merely mathematical differences in commensurable motions of commensurable elements. If however we look more closely at these cases, we find the fact to be, not that our sensations, a and b, are reduced to motions, α and β, commensurable with one another and with the sensations, but merely that the occurrence of α or β and its effect upon us is represented as the condition upon which the sensation a or b necessarily arises. The perceived colour a remains just as incommensurable as ever with the vibration of ether, α, by which its origin is explained; and if experience did not teach us that a is the consequence of α, we should have no logical means of divining from a the nature of its cause α. What therefore science does in these cases is really to connect incommensurable elements in a way which allows us to conclude from one to the other. The original proposition that a and α, b and β, do thus mutually point to one another, is due, as I said, to experience; in deriving it from facts the laws of thought are doubtless applied, but there is no special form of thought involved such as could solve the insoluble problem of making commensurable what is really incommensurable. But when experience has informed us of the coherence of two such elements, a and α, then thought concludes that this coherence will be maintained even in the event of their both changing, and that therefore a definite change of α into α^1 must always be

answered by one and only one definite change of a into a^1. Again, these changes themselves, $a-a^1$ and $\alpha-\alpha^1$, are not directly commensurable, either in kind or amount: if the number of vibrations of the sound-wave is increased by the amount $\delta=\alpha-\alpha^1$, it is true that a definite increase, $d=a-a^1$, in the heard tone depends upon it; but this change in the pitch is a process quite different in kind from the increase in the number of vibrations, and cannot be compared with it; each quantity can still only be measured by a standard of its own, and their mutual coherence can be expressed as a fact and nothing more. But the changes in pitch are commensurable with one another, and so are the changes in the number of vibrations; and if we refer these changes to d and δ as their respective units, we may ask, By how many units m of the kind d does the pitch change, if the number of vibrations changes by μ units of the kind δ? m and μ then stand in a purely numerical relation. This relation may be infinitely various; but, as before, I shall not indicate the possible variety any further in the form which I give to this inference. I choose for its name and scheme the simplest form of proportion, $E:e=T:t$, which, though it only illustrates the case in which $m:\mu$ is a constant quantity, still sufficiently symbolises the logical idea implied in the process.

114. I will illustrate that idea once more by a very elementary example. Two angles E and e are commensurable; so are two segments of a circle T and t; but an angle and a segment are incommensurable and cannot be directly measured by any common standard: so too the difference of two angles, which again represents an angle, is incommensurable with the difference of two curves, which again forms a curve. Nevertheless, if it is once established that a certain length of curve t belongs to an angle e at the centre of a circle of a given diameter, and if we form the angle E by m times e and the corresponding curve T by n times t, then the pure numbers m and n are

commensurable, which tell us how many times the two intrinsically incommensurable units t and e have to be multiplied in order to find two corresponding members in the two series of angles and curves. For the circle geometry tells us that $m = n$. Given therefore the two units, e and t, we only require to know a definite number E of e in order to arrive at the proper value of T by the proportion $E:e = T:t$. Expressed as a syllogism, then, the whole process would answer to the scheme,

$$\begin{array}{rl} \text{Major premiss:} & E : e = T : t. \\ \text{Minor premiss:} & E = F(e). \\ \hline \text{Conclusion:} & T = \dfrac{F(e) \, . \, t}{e}. \end{array}$$

115. I need hardly point out that upon this *inference by proportion*, in the simple scheme of which I include all more complex relations between m and n, rests ultimately the whole possibility of bringing qualitatively different occurrences into such mutual dependence as allows us to calculate one from another. It is also scarcely necessary to observe that we can only expect this figure to be fully effective, so far as we succeed in reducing the relations of things to terms of pure quantity: we should justify this limitation in the same way as we did the similar limitation of the syllogism by substitution. In a more lax way we are constantly judging of things, even in ordinary life, on the ground of inexact proportions, which mostly pass into mere comparisons: a general likeness is found between the relation of a to b and that of α to β, but the equal exponent of both is not precisely specified, and so the inferences drawn generally carry little conviction; e.g. 'If one of these relations under a certain condition c has a certain result γ, the other will have a generally similar result under the same condition.'

I have only one more remark to add, in repetition of what I have already said, viz. that the form of proportion

indicates a limit of knowledge. We find in it the interdependence of two members *E* and *T* merely expressed as a fact, and as such utilised for further purposes; on the other hand, the question remains unasked and unanswered, in what way, by what means, through what mechanism, so to say, the one member *E* sets about bringing the other *T* into any sort of dependence upon itself, especially into this particular sort. Of course there are a great many composite phenomena, in the case of which this question too can be answered: scientific investigation, as we said, has reduced many pairs of apparently disparate properties or occurrences to merely quantitative differences of commensurable terms, and we are then able to see how it comes about that *T* must be connected with *E*, and a particular increase of the one with a particular increase of the other. But there is a limit to this possibility: the ultimate discoverable laws of phenomena will always be found to involve determinate relations between disparate elements, which we can only accept as facts and utilise in the form of proportion, without being able to show the reason why the two elements must be proportionals. We refer many phenomena to the law of gravitation, the intensity of which is reversely as the square of the distance; but hitherto at any rate no attempt has succeeded in showing how the distance contrives to weaken the force. We show how the sensible pitch increases with the increasing number of vibrations, and how our sensations in general, and in fact all our mental activities, change proportionally to physical motions in our organs; and yet after all, tones and vibrations, mental functions and physical motions, remain for ever intrinsically incommensurable, nor do we ever experience how the one contrive to compel the others to corresponding changes. From one disparate thing to another our thought has no means of transition; all our explanation of the connexion of things goes no further back than to laws which admit of being expressed in the form of

proportion; and these laws make no attempt to fuse the two elements into an undiscoverable third, but leave them both in their full difference, and merely point out that, in spite of their mutual impenetrability, they come as a fact under a common law by which they mutually determine one another.

116. In the actual application of the inferences from proportions another defect, hitherto only briefly indicated, is tacitly supplemented by attending to an idea which necessarily accompanies them; this supplementary idea we have now explicitly to recognise as having a place of its own in the systematic series of intellectual operations, the last place in the present group. In the above scheme the proportion between the changes of two marks E and T was represented as if it always subsisted between the two marks as such, it being indifferent in what subject they occur. Now there are, it is true, predicates which upon logical grounds, on account of their contrary or contradictory opposition, or because the one in any case includes the other, must be either present together or absent together in every subject: but there are no marks whose quantities and quantitative changes must always stand in the same proportion to one another, whatever be the nature of the subject in which they are united. On the contrary, it is just this nature which determines the exponent of their proportion; and the same universally expressed marks E and T, which in one S can only coexist in the ratio $n : m$, are in another S^1 only possible in another ratio $n^1 : m^1$. Heat expands all bodies, but the ratios of the degree of expansion to an equal increase of temperature are different in different bodies. In practice, where we always have to do with individual subjects, and have these in mind throughout, we do not need to state this limitation expressly; but logic is bound to emphasise the fact that only on the assumption of the limitation can we talk of using proportions. Nothing but the specific character of a given subject, in obedience

to which all its marks mutually determine one another, justifies us in concluding from a known value of one of them to the corresponding value of another according to a proportion which holds good for this subject only. This merely brings us back to the idea which lay at the root of analogy; for it was only on the strength of the coherence of all mutually determined marks in a concept, that we felt justified in inferring from a limited group of them to the necessary presence or absence of others, as we might infer from the beginning of a pattern to its continuance. This tacitly assumed condition must therefore be added in order to complete the expression of the proportional syllogism, and its major premiss ought to stand thus, 'If S is an M, for this S it is always true that $T : t = E : e$.' And the problem which logic presents to us would not be merely to establish this major premiss through experience, in order then to bring a particular case under it in the minor, 'S is M'; it would rather be to show how a concept M can be found at all, such that the proportions required between every two of its marks can be derived from it.

117. The means for the discovery of such an authoritative or constitutive concept have already been indicated; they lie in the fact that every mark is determined throughout by every other, though in very various ways. The effect of this variety will be that, while in certain cases the presence of a single proportion between any two marks is sufficient to determine the rest, in others the knowledge of certain essential marks is necessary in order to deduce the unessential from them, but knowledge of the unessential is not enough to establish with certainty the whole content of the concept. But I shall be clearer if I preface these reflexions by an instance of the actual realisation of our requirement in the shape of a very familiar and simple mathematical form of thought. Analytical geometry possesses in the *equations*, by which it ex-

presses the nature of a curve, just that constitutive concept of its object which we are looking for. A very small number of related elements, the indeterminate abscissae and ordinates in their combination with constant quantities, as constituting a primary proportion, contain, implicit in themselves and derivable from them, all relations which necessarily subsist between any parts of the curve. From the law expressing the proportionality between the changes of the ordinates and the abscissae every other property of the curve can be developed, its course, its openness or closedness, the symmetricalness or unsymmetricalness of its parts, the uniformity or measure of alteration in its curvature at every point in it, the direction of its concavity or convexity, the area which it contains between any given limits. It is in view of these developments (the further course of which is too simple to need mentioning here) that we give the name of *inference from constitutive equations* to the method in question. The method itself is not confined to these geometrical problems; but the other and in some ways much more interesting examples supplied by other branches of mathematics, especially the calculation of variations, cannot be so easily represented with the simplicity requisite to symbolise the form of thought which we are considering. Natural science also could furnish approximations at any rate to what we are looking for. Chemistry would possess constitutive equations for analogously compounded bodies, in which the different chemical elements take the place of co-ordinates and constants, if it could succeed in expressing by its formulae not only the quantitative proportions of the elements, but also, more exactly than its symbols at present do, the rule for the grouping of atoms and the general character of their interaction.

118. Admitting the objection to the whole of this method, that like the preceding one it is not fully effective except in mathematics, we rebut in the same way as we did before

the censure which it seeks to convey, and only examine it more in detail with a view of finding new ways to supplement what is still defective in the method. It is true that the apparent wealth of development from geometrical equations is, from a logical point of view, more specious than real. In order to determine the form of the curve we give one of the co-ordinates x arbitrary values, calculate the corresponding values of y from the equation, and then connect the extremities of the perpendiculars (y) erected upon the extremities of the abscissae (x) so as to form a continuous line; the curve is therefore only the geometrical locus in which the countless results of a countlessly repeated proportion between different values of the co-ordinates are combined. As for all the new properties which we proceed to deduce, concavity, uniform or varying curvature, closedness or openness, falling or rising of the curve to this or that side, though at first they look like new marks, they also are really nothing but relations of magnitude and position between spatial constructions, relations, it is true, between different elements, but otherwise of the same nature as those assumed between the co-ordinates. Starting with a proportion between two marks x and y, we do not arrive at really new marks, qualitatively incommensurable with the first; we advance merely from given homogeneous relations to new homogeneous relations, and the derivability of the latter from the former, as well as their apparent novelty, depends merely upon the nature of space and upon the rules which geometrical perception has followed in reducing spatial relations to the universal laws of arithmetical quantities. These inferences therefore are far from meeting our requirement. The case is very different when we have to deal, not with mere spatial magnitudes, but with concrete objects, in which a number of qualitatively incommensurable marks are united, and in which moreover science is unable to explain these primarily incommensurable elements as merely different combinations of commen-

surable ones; in the face of these difficulties, thought will still have to look for a form which promises, approximately at any rate, the same advantages as those which mathematics with its easier problem offers in full.

119. The group of mathematical forms of inference ends naturally here, with the emphatic recognition of the fact that the point which does not admit of being dealt with mathematically, the disparateness of marks, is precisely the point which we cannot avoid considering. The place of the equation will be taken externally by the form of definition, for this combines a number of heterogeneous marks into a whole, but distinguishes in them a group of essential from another of unessential ones; the former are regarded as containing the law for the combination of the whole, the latter as dependent on and determined by the former in accordance with that law. Lastly, this privileged group of essential marks can only be found by a comparison of the given concept with those which resemble it, and thus we are driven to systematic forms of grouping different things, and, primarily, to *classification.*

C. The Systematic Forms. Classification. Explanatory Theory. The Dialectic ideal of Thought.

120. When we began the account[1] of the formation of our concepts, we were already at the opening of the road which we have now to travel. We already recognised the matter of an idea to be a totality of different marks, united according to some definite rule that governed their connexion; we already expected to find such a rule only in a group of marks possessed in common by different but comparable ideas; and already we noticed by anticipation the ascending scale of higher and higher concepts which results if we continue this process of comparing that which admits of comparison. I say, 'by anticipation,' because the suggestion then made has not so far been turned to account

[1] [Sections 20–33.]

in the later developments of logical activity. Judgments and syllogisms based on subsumption have only required us to consider the one relation which obtains between a concept S and its proximate higher universal M; there was no occasion for following up the relations of M itself to the higher grades of the series of concepts above it. For our only object was to make sure that a predicate P, which, for whatever reason, belonged to an M, must also belong to every S that falls within that M, and for this purpose the logical structure of M itself was to a great extent a matter of indifference. As middle term it bore the name of concept, but the character of a concept was in no respect essential to it; any simple mark, any sum of marks, whether combined under a definite rule, or merely brought together anyhow in thought, was good enough to constitute such a middle concept. It was only our concluding reflexions, which I shall not recapitulate here, that drew our attention to the necessity that the middle term should be a *concept* as we understood it at first, if we are to derive from it the right and obligation of a subject to possess the marks that it displays; for it is only when thus understood that the concept really forms the complete rule under which the whole content presented by the subject coheres and is organised.

121. In saying this we are not simply returning to an earlier standpoint. In considering the most primary and simplest forms of thought, the logician can as a rule only elucidate their results by the use of examples which contain more logical work than he means them to illustrate. For these examples must be drawn from language; and language is not the expression of a thought which has stood still where it began, but of the developed thought which has advanced by a multitude of successive steps beyond the imperfect results of its earliest endeavours, and which now conceals the recollection of them under the more elaborate setting which it has now given to its objects. And it may

therefore seem as if our present problem, the formation of an essential concept, had been solved already in the above-mentioned passage; but it needed more than the logical acts which were then under discussion to generate the ideas which were there employed as instances; such ideas could only arise by help of the processes which have now, familiar as they are, to be considered in their place in our system. Thought, in that earlier stage, met the countless multiplicity of composite images presented by perception, on the one hand with the desire to grasp each individual image as a whole whose parts are connected under a definite law, on the other with the consciousness that such a law could only be discovered by the comparison of many comparable individuals and the retention of the common element in all. But such a comparison depended for the value of its results on one condition, namely, that the attention which executed it should be directed to a number of objects *S*, *R*, *T*, whose common element really consisted in the pervading law of their whole structure, and not to a number of others *U*, *V*, *W*, differing in all respects except the possession in common of a limited group of marks. Then, in the beginnings of thought, there was no logical rule for this selective guidance of the attention; on the other hand, it was even then most effectively secured by the psychical mechanism, which makes those compound ideas reproduce one another predominantly in memory which are similar in the whole form of their connexion, and specially commends them to the attention, to the exclusion of those whose structure is dissimilar and whose agreement is confined to isolated groups of marks.

122. In the actual course of its development, therefore, thought is first directed to those universal concepts which really contain the law for the complete formation of the individuals for which they are required; it is not until it has some special motive in investigation that it frames universals in which things otherwise unlike are grouped under a fraction of similar elements. Thus when we were speaking of the

first formation of concepts, the current instances of subordination, e.g. of Caius and Titus to the concept of man, or of the oak and beech to that of plant, seemed to us quite natural and intelligible; it was as if the mere direction to grasp the common element in the individuals was enough to put us upon the track of these really authoritative concepts *M*. And yet the same direction might equally well have led us to invent for negroes, coal, and black chalk a common name *N*, expressing the union of blackness, extension, divisibility, weight, and resistance: only the tendencies of the psychical mechanism favoured the first and hindered the second of these applications of the logical rule.

123. These tendencies, which have hitherto unconsciously put us on the right way, we have now to translate into logical activity; in other words, we have to become conscious of the reasons which justify us in setting up a certain universal *M* exclusively as the authoritative rule for the formation of a number of individuals, instead of some other *N* to which we might have been led by comparing the same individuals upon a different principle. Logic has shown us that a single form of interdependence between several related points gives rise to different results; we saw that the truth of the particular followed from that of the universal, but not that of the universal from that of the particular: and that while we could always infer from a definite reason to a definite consequence, a given consequence need not always lead back to only one reason, but might lead to several equivalent ones. Applying this to the organisation of a concept, we find in it certain marks $a\ b\ c$ the presence of which has a determining influence upon the presence, absence, or modification of others, while the presence of these others, $\alpha\ \beta\ \gamma$, does not necessarily affect the former, but is equally compatible with different ones, $p\ q\ r$. This is the ground for the difference already mentioned between *essential* marks, $a\ b\ c$, and *unessential*, $\alpha\ \beta\ \gamma$; it is only in the union of the former that we could expect to find the

authoritative concept for the individuals compared, for it is only this union which determines the other marks and therefore includes none but those individuals which are of kindred structure throughout; the latter group of marks, on the contrary, would leave the former undetermined, and would therefore, if conceived as a universal, comprise a number of individuals otherwise entirely different.

124. Our problem accordingly would be, to distinguish the essential marks from the unessential. This is easy so long as we have to do with objects which we can observe in different circumstances; in that case the variable properties, which come and go as the conditions change, contrast of themselves with the permanence of what is essential. It is different when there is no possibility of such observation, and where, in the absence of varying circumstances, our object is to separate the essential from the unessential in permanent and invariable marks of the same concept: we have then to substitute comparison of different instances for observation of changes. Suppose $a\ b\ c\ d$ to be the group of marks in one case of a given concept; then, if in a second case of it d is wanting or is replaced by a quite different δ, it follows, on the assumption that all the parts of the concept cohere, that the remaining marks also experience a change; I denote the second case by $a^1\ b^1\ c^1\ \delta$, to indicate that the alteration of d to δ does not cause the entire disappearance of any one of the marks in their universal sense, but only the transition of each from one of its possible modifications into another, the form of their combination remaining the same. In this case d does not belong to the essential marks; it is the group $A\ B\ C$, including as modifications $a\ b\ c$ and $a^1\ b^1\ c^1$, which regulates the organisation of the concept. But this first step informs us only that the marks united in $A\ B\ C$ do as a fact remain together; it does not show what internal coherence they have; the value of the several elements of the group may be very different; it is possible that only $A\ B$ or $A\ C$ or $B\ C$

contain the real law for the formation of the whole, while the third mark is merely a necessary sequel or allowable addition to the other two. As the mind is not yet in a position to investigate the actual object with all the appliances of science, its only method of deciding this doubtful question is to continue the same process. We must compare $A\ B\ C$ also with instances of the form $A\ B\ T$; if the difference of the last mark is here too accompanied by no more than the previous deviation in the others, and the connexion of the whole remains the same, the coexistence and relation of A and B will be the dominant rule for the original $a\ b\ c\ d$, or will represent that union of essential marks which makes the presence of the rest possible or necessary, or at any rate determines their amount, connexion, and relation to the whole. If we conceive this process continued, we find ourselves on the way to *classification*. We can now no longer confine our consideration to the individual if we would determine its concept; that can only be done in this first of the *systematic forms*, that is, by investigating its nature in its relation to others, and judging from its position in an ordered series what degree of formative influence its several marks exercise upon its whole nature and behaviour. The authoritative principle of its formation will appear to us to lie in that inner circle of marks which, when we ascend through the next universal to higher and higher degrees of universality, remains together the longest and unchanged in its general form; and the only way to conceive completely the nature of the particular is to think of this supreme formative principle as being specialised gradually, in the reverse order to the grades of universality, by new accretions which come within the influence of its reaction.

125. The desire to get an explanation of the inner structure of the composite object by this systematic arrangement, lies at the root of all scientific classification, but is not equally satisfied by every form of it: before

going on to consider the only form which will serve our purposes here, I will therefore briefly mention, as a preliminary, the *artificial* or *combinatory* classifications, which are designed specially to meet the general demand for clearness and summarisation, or certain particular requirements of applied thought. We first by partition break up the content of a given universal concept M into its universal marks $A\ B\ C\ldots$, and each of these by disjunction into its various modifications which cannot coexist in the same subject, A into $a^1\ a^2\ a^3\ldots$, B into $b^1\ b^2\ b^3\ldots$, C into $c^1\ c^2\ c^3$. Then, on the principle of the disjunctive judgment, every species of M must possess one modification of each of the universal marks of M to the exclusion of the rest. If for the sake of simplicity we confine ourselves to two marks, of which the one, A, falls by disjunction into only two members, a and b, the other, B, into three, α, β and γ, the binary combinations arrived at in the ordinary way, $a\,\alpha$, $a\,\beta$, $a\,\gamma$, $b\,\alpha$, $b\,\beta$, $b\,\gamma$, will comprise all conceivable species of M. Lastly, it makes the collective survey of them more easy if we place the modifications of the particular mark which forms the basis of classification before the other marks, as was done above, or in the form $M=a\,(\alpha+\beta+\gamma)+b\,(\alpha+\beta+\gamma)$. The simplest instance of this classification is the arrangement of dictionaries; the fixed order of the letters in the alphabet here gives the basis of division, not only in the first instance, but also for the numerous subordinate combinations contained under the head of each letter. The obvious advantage of this lexicographical classification is, that it gives a survey of the material, not only embracing all the words of the language, that is, all members of the object to be divided, but also making them easy to find, and this first advantage it shares with all successful attempts at artificial classification; but when we go beyond this we find that the degrees in which they contribute to the real knowledge of their objects are very various.

126. We observe firstly that this method of combination only takes account of the marks of the given concept in their isolation, not in that mutual interdependence in which alone they really constitute the concept. Thus it is true that the sum of the combinations discovered includes all species of *M*, but it may also include others besides them, which would be true species if the concept were merely the sum of its marks, but are not true because it implies their union in a certain definite form which these other species contradict. The concept of a triangle does not consist in the fact that we think three angles *and* three sides, but in the fact that three sides intersect one another so as completely to bound a plane space and by this very fact produce the angles. It is this connexion of the sides and angles which makes equiangular unequilateral and rectangular equilateral triangles impossible: in a classification by mere combination these would have found a place along with the equiangular equilateral, the rectangular isosceles, and other possible kinds. If the content of *M*, as in this instance, is completely known and can be exactly constructed, these impossible forms are excluded by our knowledge of the fact, and the only use of including them in a provisional classification would be to stimulate attention to the nature of *M*, and to the reasons which make the valid kinds possible and the invalid impossible. If on the other hand *M* is a generic concept derived from experience, the inner organisation of which can only be represented imperfectly by description, not exactly by construction, the species which we have not actually observed but should have been led to infer by the method of combination, remain doubtful; further observation may discover them, further knowledge of facts may show them to be impossible; the use of assuming them provisionally may here also be to stimulate advance in one of these two directions.

127. If the method of combination, when applied to

objects of experience, is liable to the uncertainty whether its results do not include more than the facts, it is true on the other side that, as ordinarily practised, it gives no guarantee that they exhaust the facts. It is beyond the power of human imagination to anticipate completely all the modifications to which a mark may be subject; our attention will always be confined to those, $p^1 p^2 p^3$, which we happen to have observed; another modification, p^m, which does not come within the circle of our experience, will be missing in our classification along with all the species in which it may possibly occur, and this gap will not be filled up until our experience has grown. This is the ground for a logical rule, which is valuable when the decision of a question involves exhaustive knowledge of all the possible cases of some object Z; the rule is to go on dividing and classifying them by simple contradictory opposition. The sum of all possible cases of Z is always of the nature Q or of the opposite non-Q; the cases of the form Q are always either R or non-R, those of non-Q always either S or non-S; so that at whatever point the division is broken off, all possible cases are included by it. Such a method, indeed, is only fruitful when we are so happy in our selection of the first opposites Q or non-Q, or of all the subordinate opposites in the same grade, S, non-S, R, etc., that we can show without much trouble whether or no the characteristic in question Z is exhibited in each of the alternative cases.

128. It is moreover evident that in classification by combination there can be no logical rule obliging us to employ certain marks at the top as bases of division in the principal groups, and certain others lower down in their subdivisions. So long as the concept M which is to be divided is considered merely as a sum of its marks, without regard to their mutual relations, any one of them has a right to form the principal division by its modifications, and any other may be subordinated to it as basis of a subdivision. The

obvious disadvantages of this uncertainty are avoided in practice by concomitant reflexion and an estimate of the different values of the marks, based upon a knowledge of the facts or a right feeling, often merely upon an instinctive taste: all that logic can contribute to these precautions is the general direction not to choose as bases of division *notiones communes*, i.e. marks which are known to occur in the most different objects without exercising any recognisable influence upon the rest of their nature. The positive direction answering to this prohibition, viz. how to find the decisive bases of division, logic leaves entirely to be given by special knowledge of the matter in question. And as regards complex concrete objects at any rate, so long as fundamental divisions were based upon single marks, the specialist has always been open to the criticism that he sometimes removes closely related species to different and often very distant parts of the system, while he brings others which are totally and strikingly unlike into surprising proximity. This is quite intelligible when we consider the different influence which the marks have on the structure of the whole concept. There is no reason, for instance, why the mark B, so long as it occurs in the modified form b, should not conspicuously affect the formation of the whole, and in that case all the species under the head of b will remain connected in form; but the same mark may entirely lose this influence as soon as it enters into the group of marks in the modified form β; then the species under the head of β follow all the variations due to the now influential difference of the other elements $A\ C\ D$, and examples of M, otherwise most unlike, now find themselves in the closest proximity. This is what happened to the Linnæan system, which selected the number of stamens as the basis of division; the result of this view was, that in the cases where the whole organisation of the plant made the stamens of importance, the related species were brought together; where this was not the case, they were separated,

and different species were united. An instructed taste will partially obviate this evil also, by selecting different bases of division for different sections of the whole system. Nothing but an unseasonable logical pedantry could require that a system which had begun by dividing its whole object-matter according to the modifications a b c of one mark A, should go on to arrange all the groups formed by a, b, or c, according to modifications of one and the same second mark B; it may be that the variations of a mark C are exclusively of importance for the group with a, and those of a fourth mark D for the group with b, and the classification which proceeds upon this view approaches by that means, and by that means only, to the real essence of the thing. The risk which such a method runs of not discovering all the species completely, must be avoided in some other way; classification does not create the complete material, but assumes its completeness to be guaranteed elsewhere.

129. Classifications would belong entirely to applied logic if they aimed at nothing more than complete summarisation, such as is required either when we wish to deal with a subject practically or when we are just beginning to consider it logically. But they do more than thus merely prepare the ground; they themselves represent a logical ideal, which has its necessary place in the systematic series of the forms of thought; the very fact that a manifold material has been brought into the connexion of a classified system, is of itself supposed to tell us something as to the nature of each and all of its members, and not to be a mere preliminary to future enquiry. This appears in the objections which we make to forced classifications; we not only require the lines along which we must look, in order to find a particular species, to be precisely laid down beforehand in a series of concepts, but we expect the actual places in which the several species are found to correspond in position to the affinities of the species themselves. For practical purposes any order will serve that is

handy for the person who is going to use it, but the order which logic demands must be true to the facts. Now if we wish to form a complete idea of any composite object, it does not matter with which of its parts we begin, provided only that the order in which we add each new part is adapted to the particular point with which we have chosen to start: any idea of a given content so arranged forms a *concept* of it, sufficient to distinguish it from others and to show what it is itself. Amongst these various concepts of the same *M* there is one distinguished from the rest by having for its starting-point the law which determines the order of all the other marks, and this is the one which we try to find. We have already given the name of 'constitutive' to such a privileged concept; it might also be called, in opposition to the mere conceptual form in general, the logical *Idea*[1] of the object, or, in the vernacular, its *thought*; for it is thus that our language distinguishes the 'idea' of plant or organism, as its formative law, from the concept of it, which merely comprises the sum of the necessary marks and the form in which they happen to be combined.

130. It will help us to realise what has just been said if we mention here two incidental notions which always attach themselves readily to this search for the Idea of an object, conspicuously in the attempt of naturalists to improve the artificial classifications of plants and animals by reference to their natural affinities. In these cases we are prone to regard the universal Idea of animal or plant as a living and operative force, whose unvarying and consistent activity gives rise to a series of different forms, accordingly as external conditions determine one or more of its points of incidence and oblige it to change correspondingly the whole course of its action. Another way in which we are equally prone to regard it is as an unvarying end, which regulates its modes of operation according to the relations in which it finds itself placed, and in the different forms

[1] ['Idee.']

which it is thereby compelled to assume realises one and the same purpose in various ways or with various degrees of completeness. From this point of view the different species classified together express the result of the interaction between the universal idea and the particular relations, with which as universal it has nothing to do. It will be admitted that these ways of looking at the matter place it before us in a clear and vivid light, but it will also be objected that they are both quite foreign to logic. The objection is unanswerable; our intention however is not to turn the ideas of active tendency and purpose to account for the benefit of logic, but to show that even in their proper place they only have meaning on the assumption of a purely logical notion, which we will now explain. If it is to be possible for the same end to be fulfilled under changing circumstances, it must also be possible to express its content by a group of ideas, *Z*, in which these different forms of fulfilment cohere as possible species, and from which they necessarily result if each one of the marks of *Z* and each of their mutual relations is successively subjected to all the changes of which, as parts of *Z*, they are respectively capable. If, again, an active tendency is to change its activity under varying conditions and to manifest itself in new results, the combination of forces in which it consists must be expressible by equations, from which all these new formations necessarily follow as soon as we give the quantities entering into the equations all the values successively which their natures allow. Activity, then, whether intentional or unintentional, never produces anything but what is abstractedly possible to thought, and this becomes necessary to thought as soon as we affirm one of a number of related points upon which the rest depend. It is this which we have in view here: we regard the idea for which we are looking, neither as the intention of a reflective consciousness striving for fulfilment, nor as an active force which causes its results, but merely as the conceived or

conceivable reason, the consequences of which under certain conditions are the same in thought as those which must follow in reality, under the like conditions, from an intelligent purpose or a causative force. Keeping this in mind, we may tolerate a phraseology which imports into logic the idea of an end or of a tendency to development: it will nevertheless be better to avoid these expressions, and not to use what is found only in the real world as a name for the mere reason upon which in thought the reality rests.

131. Another point which logic cannot neglect may be introduced here as a sequel to these accessory notions. We are not surprised in a self-realising tendency if, under certain conditions, it fails in its endeavour; and we find it intelligible that an end should be attained under different circumstances with different degrees of completeness. Thus both these notions very naturally give rise to the assumption that different realisations or examples of the formative idea are of different values, and that they are not merely co-ordinated in a general way as species under the universal concept of their idea, but form within this co-ordination an ascending or descending scale in which each one has its uninterchangeable place between certain others. The attempts at natural classification, which endeavour to satisfy our modern requirements, are dominated throughout by this thought; and it remains to show that this familiar tendency to pass from classification by mere combination to classification in the form of a developing series, is justified on general logical grounds, and that this is the place to justify it.

If, as is too often the case at the beginning of logic, we regard a concept *M* merely as a sum of marks universally expressed, there is no sense in rating one of its species higher than another. Every *S* either contains all the marks of its universal *M*, and in that case it is a species of it, or it does not contain one or other of them, and then it is, not an imperfect species, but no species at all of *M*. But living

thought in actual practice is far from acquiescing in this hard antithesis; it distinguishes species which correspond or are adequate to their generic concept in various degrees. The possibility of making this distinction depends primarily upon quantitative measurements to which the several marks and their relations are possibly or necessarily accessible. The structure of generic concepts, incalculably as it varies in particular instances, agrees in the main in containing a number of parts or related points, each comprising a group of simple marks and standing to the others in all sorts of relations. By 'simple marks' here I mean, not only sensible properties such as red, sweet, hot, but others also like heavy, extended, irritable, which, though no doubt they contain the result of previous observations of complex modes of behaviour, contain it in so simple a shape that our logical imagination has long accustomed itself to attach them to their subjects as stable and simple predicates. To all these elements of the concept quantitative differences extend. No mark of any one of its parts is conceivable without a definite degree of its specific kind of intensity, and the degrees may vary infinitely; the number of the parts themselves can, like every number, be increased or diminished, and every part moreover can alter its logical value by expanding the simplicity which belongs to it as a member of the genus into a complex organisation of its own inner nature; and lastly, every relation between the various constituents of the concept varies in value according to the value of those constituents, or admits of greater or less closeness according to some standard of its own. The joint effect of all these possibilities of variation is to produce a number of species noticeably different. If we suppose that when a mark P of the generic concept M assumes the value p, the influence which it always exercises upon the other marks is so intensified as entirely to change the form of the whole content of M, the resulting species will no longer be a species of M, but of some other genus N. And

those values of P which approach this decisive limit but do not reach it, will produce forms which still fall under the genus M, but approximate gradually to the structure which is characteristic of N. It is upon this that the difference is based between species which are more and less appropriate or adequate to their common generic concept; each species is in a certain respect more perfect the farther it is from passing over into another genus, and that is the logically most perfect whose divergences from all proximate genera make up the greatest total amount.

132. I believe I am justified in saying that this point of view belongs entirely to logic, and is independent of the views which we may form on other and material grounds as to the value, meaning, and function of anything which has the law of its existence in a generic concept. I will therefore illustrate it by examples which are not affected by these incidental considerations. The equation of the ellipse, $a^2 y^2 + b^2 x^2 = a^2 b^2$, leaves the two axes a and b to be chosen at pleasure, and the formula claims that it will always produce an ellipse whatever values we may assume for a and b, and even therefore if one of them be assumed to $= o$. But in that case the curve passes into a straight line, and the result which this value gives falls accordingly under the concept N, that of straight line, which is different from that of the ellipse. But this example shows at the same time, what we did not choose to assert universally above, that the extreme species of a genus M, when produced in this way, not only must belong to a new genus N, but may also continue to come under the former genus M. It is true that the central equation of the ellipse can tell us nothing about this case when $b = 0$, because it then ceases to indicate a curve. But there is another expression of the essential formation of an ellipse which is still valid; namely the rule that the sum of the *radii vectores*, drawn from two fixed points on the major axis to one and the same point on the periphery, is constant and equal to the major axis. In

the present case where the ellipse has shrunk into a straight line the two extremities of the line are identified with those two fixed points, the *foci* of the ellipse, and for every intermediate point *c* we have the sum of the distances $ac + cb$, that is, the sum of the two *radii vectores*, equal to the length ab of the straight line.

If a heavy rod of the fixed length ab stands with one end *a* on a perfectly smooth horizontal surface, and with the other *b* leans against a perfectly smooth vertical wall, the pressure of its weight makes equilibrium impossible and it falls. An easy calculation shows that the path described during its fall by any point *C* in its length is an ellipse. At the same time it is clear that the end *b* must slide down the wall in a straight line perpendicularly, while the point *a* must move away upon the smooth surface in a straight line horizontally. As then every point in the line is affected by the same group of conditions, these rectilinear motions also must be regarded as specific forms of the elliptical path required generically by those conditions. They are in fact the two extreme cases which we get if we make first one and then the other axis $= 0$; the end of the rod then moves in a straight line in the other axis. The middle point of the rod supplies another singular case; the axes of its elliptical path are equal, and thus it describes the arc of a circle. The nature of the problem before us compels us therefore to conceive the circle as a species of ellipse, and the central equation which we have mentioned makes it at once clear how this is possible. This example therefore shows us that by changes in the quantity of one of their parts the species of a genus *M* approach gradually to the formative law of another genus, and that there may be limiting instances which are species both of *M* and of *N*, because they satisfy the requirements of both concepts; by merely examining the actual constituents of such a limiting instance it is impossible to tell by which generic law its form is, strictly speaking, determined; in the present state

of our knowledge this question is decided upon incidental grounds of various kinds.

133. On the other hand, these examples leave an ambiguity which must be removed in regard to the standard by which we measure the degree of perfection, or, to put it shortly, the height of each species. Mathematical figures have no history telling of their life and growth; being merely legitimate possibilities of thought without real existence, they can be produced for our imagination in the most various ways, and it is in the abstract indifferent, and in any particular case depends on the nature of the problem in question, from what point we begin their construction, or under what generic concept, what universal rule of construction, we bring them. If we look at them, not geometrically, but aesthetically, I mean if we attend to the total impression of the figure as it is, not to the way in which it came into being, circles and straight lines contrast decidedly with ellipses. In the *impression* of the ellipse as we perceive it the inequality of axes is a necessary element; on the other hand it is true that the greater this inequality is, the more does the curve approach the extreme forms which we wish to exclude, that of the two straight lines which coincide with one or the other axis. The characteristic impression of the genus would be best produced by an ellipse equally removed from the equation $a-b=0$, that of the circle, and from the equation $a-b=a$, that of the straight line. By combining both equations we might define the condition of this impression by saying that one axis must be double the other, and this would be tolerably correct; only that a thing cannot be mathematically determined which does not depend simply on mathematical laws. Our logical imagination is dominated in every direction by similar tendencies. Nothing is commoner than for a person who speaks of a quadrangle to mean really a parallelogram, or often even a square; and this inexactness in expression is very natural; the imagination wants to realise the concept

in perception, but can only hold one image at a time, and it therefore chooses the image which is logically most perfect; and it is the fact that the parallelogram, by increasing inequality either of the sides or of the angles, continually approximates to the ultimate form of the straight line, in which all the four sides coalesce. The observation of natural objects evinces the same tendency; we always regard as the typical and most expressive examples of each genus those species in which all the marks are at the highest value which the combination prescribed by the genus allows, in which therefore no mark is exclusively prominent and none is reduced to zero, but all combine, as far as possible equally, to produce the impression of stable equilibrium in the whole.

134. I will here repeat an observation which I made before. I am not afraid that anyone will criticise this mode of estimating the relative height of species on the ground that it has nothing to do with logic; its defect is rather that it starts from inadequate logical grounds, and does not adapt itself sufficiently to the nature of its objects. To put it shortly; that the highest perfection of a species depends upon the equilibrium of its marks as described above, is the opinion to which we must come on purely logical grounds, so long as we have no positive knowledge to supply us with some *other* standard of measurement based upon the essential characteristics of the genus in question. It may lie in the nature of things that a genus *M* can *not* maintain this equilibrium of marks, but is destined by diminishing one and intensifying another to pass over into another genus *N*; in that case its species will be more perfect in proportion as they approach more nearly to this point of transition at which they cease to belong to their own genus. We find that the most important attempts at natural classification are deeply imbued with this idea of a destination to be attained, which is constantly impelling the several genera to advance beyond themselves; I therefore introduce it here

intentionally, in order to notice its significance for logic, with which in itself it has nothing to do. We have already[1] separated the idea of productive activity from the concept of tendency, and the idea of purpose from the concept of end; we must in the same way separate here the idea of *obligation* from the concept of destination. Everyone will see that the effect of this separation is to do away with all that is characteristic in the meaning of these three concepts; but this is just what we are aiming at. It is not the concept of destination itself which we are importing into logic, but merely that of the logical relation upon which it is essentially based, and of which it is itself so graphic an illustration that we can hardly avoid the term as a figurative expression of the logical truth. A destination, then, which has to be reached, differs from a final state which merely happens to be reached by some process of change; in the former case the group of marks which characterises the end attained contains also the authoritative principle upon which the marks are connected and upon which they change as they do; in the latter, the processes which lead to the end may take various directions, forwards and backwards, to this side and that. Bearing this in mind, we can no longer doubt as to the purely logical sense of the word when we speak of a 'destination' to which the several genera have to approach. Hitherto we have looked upon the generic concept M as the ultimate authoritative principle which regulates the series of its species, and that species therefore as the highest which exhibits this concept in the most perfect equilibrium of its marks; now we are reminded by a consideration originally foreign to logic, that the case may be different, and that the formation of the series of species in M need not really depend on anything in the generic type of M itself, such as could be discovered by merely examining its own constituent marks; that, on the contrary, the formation of this genus is not rightly explained until we compare

[1] [Above, § 130.]

it with another genus *N* into which it passes, and with a third *L* from which it came by a similar transition, and these again with those which went before and came after them; not till this comparison has been made do we get the direction in which the progress towards perfection takes place within a higher genus *Z*, of which *L M N* are species; then, in the series of species in any particular genus *M*, those species will be the highest which have advanced the farthest in the direction in which *M* as a whole is developing towards the most perfect expression of the higher *Z* which includes it. It remains to show that this line of thought, to which we were originally led by an extraneous suggestion, has its necessary place here in the internal economy of logic.

135. It is scarcely needful, however, to show this. We have seen that we could only produce the universal concept, which includes a number of individuals under it, by uniting their permanent and common marks; then we saw that this constant group of marks might contain elements of very different values, and in order to separate those which are not only constant but contain the rule to which the rest must conform on joining them, we had to compare the universal already found with other universals, and species with species; that which still cohered in this wider field of change we regarded as the true essence of a genus *M*, the species of which were to be ranked higher or lower in proportion as they realised it more or less perfectly. But this process has no natural ending; the same questions continually recur; the marks which constitute *M* will themselves differ in value, and the only way to distinguish the essential from the unessential will be again to compare *M* with *L* and *N*, to form the higher genus *Z* from the law which persistently governs the formation of them all, and to measure the value of *M L N*, as well as that of their several species, by the degree in which they realise this law *Z*, instead of by the degree in which each species expresses the

more special law of its own proximate genus. This progress might go on to infinity, or to the point at which we succeeded in finding a highest ideal *A*, exhibiting the mode of connexion to which all kinds of existence, real and thinkable, must conform: from this *A* a classification might be derived in the form of a development which evolved from itself the whole content of the universe, and this development, if it were possible, would give the only logical security that every species had a place in the series of cognate species answering to the degree of essence which it expressed. Thus the problem of natural classification leads of itself beyond the isolated treatment of a particular problem to the systematic organisation of the whole world of thought. And this tendency has in fact guided the most important attempts at such a classification. Those who have wished to exhibit the development of plants or animals in an ascending scale, or the events of history (for this form of thought claims to apply to processes also), have always been obliged to justify their selection of a particular standard for measuring the increase in value of the several members of the series; this justification they have always had ultimately to find in certain general views as to the meaning of all being and process, views which are either formally expressed at the very beginning of the enquiry, or make themselves tacitly felt throughout it as a guiding principle.

136. Natural classification, then (to sum up under the traditional name the procedure just described), differs from combinatory or artificial classification in taking account of the mutual determination of marks which in the latter received only subordinate attention, while in its result it is distinguished by its *serial* form, in which the members are not merely placed side by side, but follow each other in a definite order leading from the province comprehended or dominated by one species into that of another: this order begins with those members which answer least to the logical destination of the whole system, and ends with those which

express in the most complete and pregnant way the fulfilment of that destination. But the simplest case here supposed, that in which the series has only one direction, is not necessarily the only one. In the first place it is conceivable that single marks in each species may vary without altering the characteristic structure of the species at all, so far at least as we can see: in that case the different instances of this species are equal in value, and the series may thus be increased in breadth by co-ordinated members without growing in length. It is also possible that, owing to different or opposite variations in several marks, a species *M* may not only pass over into one proximate species *N*, but branch out into several, *N*, *O*, *Q*, with which it has equal affinity and which contribute equally to carry out the general development; these will then become starting-points for new series, which either continue side by side or subsequently coalesce again somehow with the central series. Thus the form of natural classification in general is that of a web or system of series; even the culminating point of the system need not be a strict unity, for the most perfect attainment of the logical destination is compatible with a variety of precisely equivalent forms.

137. As the occasion suggests it, I will mention two more concepts in frequent use, which may find a logical explanation here. The new kind of value which each species acquires in proportion as it approaches the end to which they are all developing, does not exclude the other kind which we mentioned earlier, depending on the equilibrium which it exhibits in the marks of its proximate [illegible]. The two values subsist side by side, though the [illegible]rs the other. We feel the conflict between them [illegible]sthetic judgment of phenomena. Every species whi[illegible]resses its genus in the stable equilibrium of its mark[illegible]esses us as perfect, relatively or absolutely: such a spe[illegible]orms the *type* of the genus, that type which is the indis[illegible]able

though not the sole condition of beauty in the beautiful, and which gives even to what is abstractedly ugly the formal right to a subsidiary place of its own in artistic representation. On the other hand, species in which this equilibrium is disturbed by approximation to an end higher than can be attained within the limits of the genus, give us the ambiguous impression which we call 'interesting,' like dissonances in music, which do not satisfy us but prepare us for a higher satisfaction. *Ideal* as opposed to *type* would mean a phenomenon in which the equilibrium of marks required to make it typical coincides happily with the highest development in regard to its logical destination; logic does not exclude the possibility of such a coincidence, and art may perhaps find it realised or be able to realise it in a phenomenon in repose, though more probably only in some situation of the phenomenon.

138. Lastly, it will be asked, how classification by development reaches its required conclusion, the certainty, namely, that it has really found that supreme law or logical destination which governs the particular object or the universe at large. To this we can only answer, that by way of mere logic it is quite impossible to arrive at such a certainty. The form of classification by development, like all logical forms, is itself an ideal, an ideal which is demanded by thought, but which can only be realised, so far as it can be realised at all, by the growth of knowledge. Nor indeed is this an exceptional condition, such as would lay this first of our systematic forms under a disadvantage. The judgment also enjoins a connexion of subject and predicate which thought has to make if it wishes to come into contact with its object in its own way; the hypothetical judgment, for instance, tells us, that only by annexing a condition to the subject S is it possible to ascribe to it a predicate P which is not already contained in the concept of S; but logic does not tell us what condition x is necessary in order to secure this particular P for this particular S; it

waits for special knowledge to put its injunctions into practice. The theory of the syllogism also teaches us how to draw conclusions when the premisses are given, but it does not give us the premisses, nor does it guarantee their truth, except so far as they may themselves be conclusions deducible from other premisses; these latter then serve as the material given to thought, and lead back finally to some truth which is no longer logically deducible. Similarly all that the theory of natural classification asserts is, that every group of complex and coherent objects, and therefore (since everything coheres) the whole realm of the real and the thinkable, must be regarded as a system of series in which concept follows concept in a determinate direction; but the discovery of the direction itself, and of the supreme directing principle, it leaves to positive knowledge to make as best it can.

139. It is not this objection, but a difficulty of another kind, which obliges us to continue our enquiry. The difficulty will be most easily understood by reflecting on the place which classification occupies in our system. As a certain arrangement of concepts, it answers primarily to our first main section, the theory of the concept itself; but we were obliged to pass on from the concept to the judgment, for we found changes in the content of thought which could not be apprehended by conception alone; on the contrary, the concept presupposed relations between its marks which it needed the judgment to interpret clearly. Classification answers moreover to the first form of judgments, the categorical; as in these the subject simply had, assumed, or lost its predicates, so here the supreme authoritative concept appears by itself as the sole producer of all its species, as the source from which they *emanate*. But the hypothetical judgment met the categorical with the objection that a single subject S cannot by itself give rise to any multiplicity; and, similarly, all theories of emanation will have to ask themselves the question, what second

condition it is which makes their first principle develope at all, and whence come the data in reaction against which it is obliged to expand into these particular forms and no others. A corresponding advance is called for here; and it will prepare the way if we consider it in still closer connexion with the characteristics of classification described above. We made it an objection to artificial classification that it may lead to impossible instances, while in classification by development we gave proportionately more attention to the mutual determination of marks; we assumed that a change in one mark reacts upon the rest, that through this change one concept passes into another, and that one species answers better than another to its concept. This clearly implies that in the formation of its species the concept depends, not only on itself, or, in figurative language, on its own purpose, but also on another power which determines what kinds of realisation of that purpose are possible or impossible, adequate or inadequate. This power we have to investigate.

140. The problems of thought are not completely solved until it has developed forms for the apprehension of everything which perception offers to it as an object and stimulus of its activity. This requirement, that *all* thinkable matter should be included, is not satisfied by classifications. Their natural objects are always those stationary generic forms with stereotyped marks, which we believe ourselves to have before us in perception as fixed points for manifold relations, but which are far from constituting the whole of what we really perceive. The several genera are not found in reality arranged in the system in which classification exhibits them; as they actually appear they are always realised in numberless individual instances, separated in time and space, and subject to continual change both in their own conditions and in their relations to one another. Even if we admit that the nature of each generic concept contains the law which every instance of it will obey *if* it occurs under

certain circumstances, yet there is no reason in the concept itself for the hypothetical addition which we make, neither, that is, for the presence of that instance at the time and place at which it is present, nor for the occurrence or non-occurrence of those particular circumstances. Thought, therefore, does not embrace in the form of classification all that there is for it to embrace; and that which appears here merely as an incidental stimulus to the universal concept to produce this or that species of itself, must also be taken account of as an essential part in the organisation of the thinkable world as a whole.

141. These considerations are not disproved by the fact that, as we observed before, classification by development may extend, not only to generic forms of the real and the thinkable at rest, but also to progressive processes. For when it is attempted to represent history as a development, the question what it is which makes process process, the coming of one state into being out of another, equally escapes the grasp of logic. When they are reflecting on the past or forecasting the future, these speculators may picture to themselves certain situations as temporary states of equilibrium, which they assume to follow one another on the stream of events in a fixed and necessary order; but how the transition from one to another actually comes about, they cannot tell us. Nor could they do so even if they undertook the endless task of dividing the interval between two such states of equilibrium into an infinite number of stages; they would be able to show that the concept of each stage, when it is reached, is preliminary to the concept of the next, but they could not show how the reality which this concept expresses brings the reality expressed by the other in its train. We must reflect moreover that in the real world pure concepts do not occur or develop themselves, but only particular examples of them, each with all its marks specifically modified in a way which its concept allows

but does not necessitate. Not only therefore does the process of becoming remain a mystery which classification cannot explain, but the result of the process results, not from the concept of the stage preceding it, but from that particular realisation of the concept of which also classification takes no account. All the attempts both of ancient and modern times to derive the world by way of emanation from an original concept, are subject to the same defect. If their original concept is really nothing but the pure thought of a relation which certain elements not yet named necessarily imply, all that they can derive from it will be certain forms, likewise universal, in the shape of possibilities, or, as I have no objection to say, necessary requirements, which in the event of being realised must be realised in a certain way; but they have no means of deciding what this way will be, or of showing where the desired realisation will come from. If on the other hand their original thought expresses a relation between elements not unnamed but definitely characterised, and is endowed itself with the impulse to development which those elements do not supply, in the shape of an inherent restlessness which drives it to evolve its consequences, this is only to admit that the complete form of each new stage of development does not depend only on the concept of the preceding stage, but on the special form in which, as a fact, but without any reason, that concept had already realised itself. It is to admit, in other words, that alongside of their categorical development by emanation of the concept out of itself, another power is also at work; this power, which their theory entirely disregards, consists of a sum of authoritative hypothetical relations, which ordain that if the marks in a given concept have as a fact a certain value, and if certain conditions act upon these marks, the form of the new resulting concept, the new stage of emanation, is then, but also not till then, completely determined. Lastly, if we compare the theory of

emanation with the method of the inferences by subsumption, we may say shortly that what it lacks is the *second premiss*, by which alone they produce from the universal major the comparatively more special conclusion. These subsidiary ideas, which are here only tacitly presupposed, logic has to supply explicitly: it cannot stop at a classification based upon *concepts*, but must point out also the legitimate connexion of the *judgments* which express the power of a mark already in existence to determine another which is to come into existence out of it.

142. But it is not necessary to confine ourselves to that side of classification where it fails to give a complete solution of the problem of thought; the attainment of its own more limited end implies the same tacit assumptions. Each of the generic concepts classified is necessarily composed of marks which occur in other concepts as well. It would be lost labour to construct a scale of genera *L M N*, if *L* had marks which were heard of nowhere else in the world, and *M* and *N* were distinguished by similar uniqueness. The marks must rather be looked upon as building-stones lying about ready for use; they have to be cut differently according to their different positions, but they are all of commensurable material, and it is only the different ways of using it which give rise to concepts of different structure. Now in classification by development the marks united in the same generic concept *M* are spoken of as mutually determining each other; a change in one is followed by changes in another; and the progress of these changes not only produces the several species of the genus *M*, but leads beyond them into the genus *N*. What rules can this influence of one mark on another follow but such as involve a universally valid relation between the natures of these *marks*? And as the marks themselves hold good beyond the limits of the particular concept *M*, this relation also must be independent of *M*. The formation, therefore, of the several species of *M*, their possibility or impossibility,

and ultimately the possibility or impossibility of M itself, all entirely depend on what is allowed or not allowed by these *universal laws* of connexion between the marks. Accordingly, the classification of concepts cannot fulfil even its own proper function without presupposing a system of judgments or universal laws regulating the admissibility, mode of connexion, and mutual determination of all marks which are to be united in this or that generic concept.

143. I must mention here an apparent contradiction, the removal of which will conclude these preliminary considerations. We have already, in treating of the form of proportion, spoken of the necessity of this mutual interdependence of marks; we there corrected ourselves by saying, that when a constant relation exists between two marks, the measure of their interaction is not found in the marks as such, but in the nature of the whole in which they occur or in the concept of that whole. We seem here to be retracting this statement, but we are in fact confirming it. For the very point which we have now made clear is, that the content of the concept, to which we there transferred the decisive influence, is nothing but a number of marks, each extending beyond the concept itself, and all connected in it in a definite way. Between these marks, as we saw, different relations are possible; it may happen that the idea of one involves that of another; in that case every subject which has the first will have the second also; or it may be that two marks exclude each other as contrary and contradictory members of a common element, and in that case there is no conceivable subject in which they can exist together; between these extreme cases lie others, in which, without any similar logical grounds, we perceive two marks to be combined as a fact, but the value of the one does not always imply a like value in the other. These are the cases to which our observation above applied; for the reason which narrows the range of this variation, and fixes the precise proportion in which two marks determine each other

in any particular object, lies in the simultaneous presence of all the other marks, in the values and the mode of their combination. What was undecided in the relation of the two is decided by their relations to the rest; if the different equations, by which we may suppose the latter relations to be expressed, are only satisfied by one value of each of the marks, the formation of the whole is completely defined; where the number of equations is not enough for this, the whole is still partially indefinite, and exhibits a universal concept in which there is still a possibility of different species. Thus it is true that the concept determines for its subordinate species the proportion in which each pair of marks condition one another; but it only does this in virtue of the *ordered sum* of its other marks, and so far as these are known to have definite values. Our method, in fact, has always been based upon this supposition. In proposing to classify a generic concept by developing its species out of it, we have always had to assume that certain of its universal marks are already defined by their places in the series; not till then could the rest acquire that definite character which was necessary to complete the distinction of one species from another. In the concept itself the existence of this primary definiteness, of which the rest was a consequence, was only a possibility; its realisation was assumed in thought independently of the concept.

144. If we sum up these considerations, we may say that every individual and every species of a genus is what it is through the co-operation of the complete sum of its conditions; these conditions consist in the fact that a number of elements or marks, which might also exist in separation, are as a fact given in a certain combination, which might conceivably be different, and each with a certain quantitative value, which is one amongst other possible values. From this given union of conditions, according to universal laws which hold good beyond the limits of these elements, this perfectly definite result follows. Every such result,

when it is once there, can be compared with others, and co-ordinated with them as species with species or subordinated to them as species to genus; but these concepts, which hitherto we are considering as the key to the understanding of the structure of their subordinates, must not be credited with any mysterious and authoritative power, beyond the fact that they are condensed expressions for a definite union of separable elements, which act and react upon each other according to constant and universal laws, and give rise in one combination to one set of results, in another to another.

145. It is evident what a revolution these considerations cause in the whole view of logic: we see it in the logical form of *explanatory theory* which modern science opposes to that of classification, by which antiquity was exclusively dominated. I leave it to applied logic to speak of the methods which this change in our thoughts necessitates in practice, and confine myself to pointing out briefly how the logical view of the world, if it were attained as these theories understand it, would differ from that of the theory of classification. In the first place, we hear no more of a *categorical* emanation of all real and thinkable matter, proceeding by the mere impulse of a plan of development contained in the point from which it starts, without the aid of any other conditions; the form of science becomes essentially *hypothetical*. It does not describe what is and what comes to be; it defines what must be and come to be *if* certain conditions are given; the question whether, and in what order and connexion, these conditions occur, is excluded from the province of logic and left to be answered by experience, which will bring the facts to illustrate the application of the theory. Nor will I here raise the question, how this theory gets at those universal laws by which it decides, that wherever a particular group of conditions is given, one particular result and no other must occur; it is sufficient at present to observe that it does start with this conception of a *law* which fixes the particular result of a

particular condition *universally*. This means, that wherever the condition $a + b$ is found, only c follows from it, and the nature of the object in which $a + b$ is found has no power to give this condition directly any other result than c; it can only do so when other conditions, $a + d$, are present in it as well as $a + b$, and the former co-operating with the latter oblige c to change into γ; and this co-operation also takes place by a universal necessity quite independent of the nature of the particular object and equally binding upon all others. And in the new result γ the law which connected c with $a + b$ is not eliminated, but continues to operate concomitantly; for $a + d$ alone would not have produced γ, but δ.

From these universal laws arises that mechanical character, of which the adherents of these theories make a boast, and their logical antagonists a reproach. The tendency to derive a series of phenomena 'organically,' as the phrase is, from the meaning of a conception which develops itself in them, is met by the assertion that a mere meaning which wants to develop itself does not produce anything, but that everything exists, and exists only, when the complete sum of conditions is given from which it follows necessarily by universal laws; it must be regarded as the result of these conditions alone, and explanation consists merely in showing that a given and perfectly determinate thing is the inevitable consequence of the application of universal laws to given and equally determinate circumstances. Animated by this logical spirit, which is found most pronounced in the mechanical sciences, explanatory theories are averse both to using and looking for universal generic concepts, and to schemes of classification. According to them a phenomenon has been merely observed, not understood, as long as it can be referred only to the special characteristics which distinguish one concept from others, and not to the prescription of a universal authority which is equally binding upon everything thinkable and everything real. It is their pride not to need generic

concepts and their arrangement in a system of classes, but to show that, whatever the context from which a phenomenon gets its meaning, we know all about it as soon as we know the sum of related points combined in it; for whatever is, is merely an example of what must come to be when the universal laws are applied to this or that particular group of given elements. Even the position which is sometimes taken up as the utmost that can be conceded on the other side, does not satisfy the demands of these theories, the position that everything obeys universal laws, but each domain of reality its own, and that the laws of living and spiritual existences are different from those of lifeless and material ones. It is indeed obvious that those special laws to which any given phenomena are immediately subordinate, and with which therefore they are most closely connected in matter and form, vary with the varieties of the subjects which they express; but there could not be two worlds depending on two supreme and independent laws, unless they had nothing to do with each other and no effects from the one were ever felt within the limits of the other: anyone who speaks of one world, embracing those different groups of self-developing things and events, must start with a single law valid for all reality, or a single unbroken circle of law, of which all the special laws of different domains are particular cases, and from which they arise as soon as it is supplied, in a succession of minor premisses, with the different conditions which differentiate the several domains of active existence.

146. In accordance with my plan of dividing the problems of logic, I have omitted from the preceding account of explanation all mention of the means which the theory employs, partly for discovering the universal laws which it assumes each coherent group of existence to obey, partly for detecting in the manifold variety of experience those inner coherences themselves which the subordination of different elements to the same common principles admits or requires.

I have reserved to applied logic the utmost freedom to follow the course of these efforts; all that came within our systematic survey of the operations of thought, of which we are now approaching the conclusion, was the form which explanation *would like* to give to the connexion of all thinkable matter, and in which, if it could really be given completely, the final goal of intellectual aspiration would seem to be attained. As to this goal itself, however, I do not share the prevailing conviction of the present day. Explanatory theory is almost the only form in which the scientific activity of our time exhibits itself; the consciousness (so late in making itself felt) of the principle which that theory has to follow, strongly separates all modern science from that of antiquity and the middle ages, and the methods of investigation developed in consequence of it form the precious treasure which places the modern art of discovery far above that of ancient philosophy. Yet the opposition so unremittingly made to this form of thought, when it claims exclusive dominion over the thinkable world, shows that the belief that it leaves nothing more to wish for is not universal. If we consider first the familiar forms which that opposition assumes in our collective view of the world, we shall be able to disengage from it the purely logical residuum of feeling which the explanatory theories fail to satisfy.

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147. The assertion that all existence is subject only to universal laws, and that every individual is nothing more than it must become according to those laws, if conditions, which might have been combined differently, have as a fact combined in a certain form, is most obviously distasteful on aesthetic grounds and to artistic natures. Beauty, it is felt, cannot be understood upon such a view; it only seems of value, and to be really itself, if the ultimate form which excites our admiration is the result of a single power, a result which is indeed inevitable, but which, besides being inevit-

able, is also the fulfilment and manifestation of a living impulse: it would appear unintelligible, if it were merely a lucky case of harmony between casually coincident elements. I have tried elsewhere to show that this aesthetic objection is wrong, if it goes on to deny the universal validity of the explanatory or mechanical theory. As understood by that theory, the meeting of the various conditions is never a matter of chance, but always the necessary consequence of the past states of the world. If we follow out this thought, it leads us back to some combination of elements which we regard as the initial state of the world; and there is then nothing to prevent us from supposing that this combination, which might conceivably have been different, contained within it the marvellous germ of beauty, which, making itself felt through the whole mechanical chain of consequences, gives birth by single acts of its own to the beauty of single phenomena. Or again, if we wish to avoid the difficult conception of an initial state, there is no reason why we should not take a section, as it were, of the world's course at any point of time that we choose, and suppose the combination of all the forces then acting simultaneously, just because it is that combination and not any other equally conceivable, to be the one and sufficient reason of all individual beauties. Such a supposition would give room for everything which our aesthetic feeling considers necessary to maintain the dignity of beauty; it would merely have somewhat changed the place of the single impelling power; this power would no longer lie self-centred in the individual beautiful thing; it would continue to be active in the individual, but only as the after-effect of a universal which permeates all individualities. By thus putting back the origin of beauty we do not run counter to aesthetic requirements; on the other hand, the mechanical theory, obliged as it is to assume some existing state of things in which the continuity of development according to universal laws is exhibited, has

no motive for conceiving that state as meaningless rather than full of meaning, as irrational rather than rational, as the source of caprice in the world's course rather than of consistent purpose. There is however one point which the requirements of aesthetic feeling and the admissions of scientific explanation equally imply, namely, that the secondary premisses, which we bring under the universal laws and by which we denote the facts to which the laws apply, cannot have the casual origin which they doubtless seem to us to have when we are absorbed in some particular field of enquiry and have taken them out of their mutual connexion. They must themselves be systematised and form parts of a whole, that whole which comprehends all real objects to which the universal laws apply. The minor premisses to our general view of the world must not be conceptions of a number of disconnected possibilities in hypothetical form, each of which, *if* it occurred, would lead by universal laws to a definite result; they ought to distinguish categorically each possibility which occurs from those which do not occur, and exhibit it as a legitimate member with a place of its own in the universal order of reality.

148. This requirement is partly supported, partly modified, by metaphysical considerations. For what would be the meaning of assuming on the one side a realm of universal laws, and on the other a sum of reality which conforms to them, if no further relation existed between the two and made this subjection intelligible? And in what could the subjection consist if not in the fact that the behaviour prescribed by the laws is from the very first an actual property of all reality, a constant mark alongside of the different or changeable marks by which one real thing is distinguished from another? No truth at any rate can be *applied*, as we are in the habit of saying, to a given content, unless the content itself answers to it; every application is merely the recognition that what we wish to apply

is the very nature of that to which it is to be applied. Now a limited number of observations enables us to discover that everything real exhibits certain constant characteristics, and these characteristics then take the shape in our mind of expectations which will be confirmed, and which we bring with us when we make further observations; thus we easily come to regard them as something which exists independently in fact as well as in our thoughts, and is prior to the object in which we shall find fresh confirmation of it; hence all that strange phraseology which regards universal laws as powers ruling on their own account, to which everything real, whatever its origin and whatever its nature, is subsequently obliged to submit. If we avoid this wrong conception, and connect that which we substitute for it with that to which our aesthetic requirements give rise, the one and undivided object in which our thought now seeks satisfaction is a being, which, not in consequence of a still higher law but because it is what it is, is the ground both of the universal laws to which it will always conform, and of the series of individual realities which will subsequently appear to us to submit to those laws. I have no intention of exhausting this subject here, and I pass over many difficulties which we shall have to notice later, some of them in the course of our present logical enquiries, others in their metaphysical context: it is enough here to follow out the logical form of thought which the mind must look for if it tries to satisfy the want just described.

149. This form will no longer be quite that of inference as described above. The universal law, to which the major premiss there gave the first place, instead of standing out from the other elements as their essential condition, will now accompany them as a latent idea, always understood but not expressed; its former place is taken by the universal nature of the sum of existence, which is developing itself in the world. Nor is this nature conceived as an ideal content at rest, which could not be set in motion without extraneous

conditions, but as the subject of a movement which enters into its very constitution and without which it would not be what it is. The particular form which the moving content assumes at each successive moment, depends on the one side upon its permanent purport and permanent direction, on the other upon its particular position or the particular point to which it has thus far developed, not through extraneous influences but through its own movement. It would be possible, but would only lead to prolixity, to express the essential truth in this kind of idea without importing into it the conception of motion; we should then find ourselves requiring an idea which includes in the system of its species and sub-species the whole of reality; but the differences and the order of these species would not be determined independently of the idea by pre-existing marks and their modifications; the idea itself would contain the reason for the presence of the marks, for their possible divisions, and for the arrangement of the resulting varieties according to their value, in fact the whole reason for its own classification. We may formulate our requirement most shortly as follows: the form of thought for which we are looking must have only *one* major premiss for all its conclusions, and this premiss must express the movement of the world as a whole; its minor premisses must not be given to it from elsewhere, but it must produce them from itself in the form of necessary and exhaustive varieties of its meaning, and thus must evolve in an infinite series of conclusions the developed reality which it had conceived as a principle capable of development in the major premiss.

150. It cannot be said that the impulse to organise the whole world of thought upon this pattern is foreign to the mind when left to itself; it has been at work at all times, and whenever a view of the world more or less like the theory of mechanical explanation has developed itself, this impulse has met it with the reiterated demand that the world and all things in it should be regarded as a *living*

development. For it is in the phenomenon of life that we believe ourselves to see these claims of the mind completely satisfied; as there the original type of the organism is made into the efficient power which produces the incentives and conditions for its own consistent development, so we would have the world as a whole evolve from itself the occasions which are the necessary conditions of its gradual self-realisation. We need not here notice the errors in this belief in the independent development of the individual organism; it is enough that it *appears* to be a graphic instance of what we are looking for. The same image has also been a constant favorite with the theory which, for the last time in our day, avowedly aspired to a vision of the universe springing out of the unity of an idea, which develops itself and creates the conditions of its progress. For it was in no attitude of investigation and reflexion, by no means of logical and discursive thinking, bringing independent minor premisses under universal majors, that the Hegelian philosophy even wished to derive the world from its single principle: it only proposed to look on and see how the development followed from the inherent impulse of the idea. And for this intellectual vision, this '*speculative*' thinking in the original sense of the word, it believed itself to have found a guide in the dialectical method, a guide which enables the spectator to follow the true course of the self-realising development. I shall still keep to my principle of saying nothing in this survey of logical forms about the practical rules for securing their application to the matter of thought, and therefore leave for a later occasion what is to be said about this method as a method; but I shall appropriate the antithesis between speculation and explanatory theory for the purpose of describing the final shape which we aim at giving to all thinkable matter, and call the *form of speculative thought* this third member, with which the series of comprehensive and systematic forms comes to an end.

151. And yet I feel that I must not conclude quite so shortly; I must return once more to an observation which I have already made. All forms of thought which we are considering are ideals; they indicate the final shapes which thought wishes to give, or to be able to give, to the matter, great or small, which it has before it, in order to satisfy its own inherent impulse by showing the coherence of all that coexists. Nor is the validity of these ideals at all impaired by the fact that human knowledge is not able to apply them to every given instance. It may be that we are not always in a position to discover the universal laws which govern a particular circle of phenomena; and it may be that, if we had discovered them, we should not succeed in bringing all particular cases under them so completely that the necessity of any given result was at once apparent. But we should not push forward our enquiries in this direction so untiringly, if we were not convinced that the principle of the explanatory theory is universally valid, and that its validity is independent of our present ability to verify it in every conceivable instance. Perhaps the form of speculative thought is in a still more unfavorable position; the conditions under which *human* thought is placed may be altogether inadequate to achieve the speculative ideal in more than a few instances, perhaps even in one; yet this ideal also will retain its binding force, and continue to express the form in which, if we could give it to the whole material of thought, our mind would find all its demands satisfied. This form also, therefore, has a right to its place in the systematic series of forms of thought: that it is the last in the series is clear without proof, for it leaves no elements remaining in mere unconnected juxtaposition, but exhibits everything in that coherence which had been all along the aim of thought. At the same time it points beyond the province of logic. From the point of view of the explanatory theory it might still seem as though the universal laws, which thought produces from itself alone,

gave a right to decide *a priori* what reality will be like; speculation does not deny this right, but by making the *content* of a supreme principle the one and only ultimate ground of everything, both of the power of these universal laws themselves, of the direction in which the world as a whole develops, and of the individual forms which in consequence reality assumes at each moment, it indicates that the final fulfilment of all logical aspiration could not be attained by new logical *forms*, but only by material *knowledge* of that supreme self-developing principle which speculation presupposes.

In concluding this account I am conscious how much its method deviates from those which are in vogue at the present day. We are so accustomed to being told the history of things, and to feel our curiosity satisfied when we have discovered or invented an origin for them, that even logic is flooded with psychological explanations and derivations of its doctrines: on the other hand it strikes us as antiquated, odd, and unmeaning if anyone attempts to arrange the forms of thought in a progressive series according to the nature of its problems, instead of following the order in which the mental activities necessary to their solution develop in the individual soul. I am content that this should be so, and hope that in the form of my exposition my readers will recognise the premonitory influence of the idealistic philosophy to which it is intended to lead: I have no fear that by choosing this form I have distorted the substance of truths which, on any view of logic, must be equally regarded as established.

BOOK II.

APPLIED LOGIC.

PREFATORY REMARKS.

152. WE are so much accustomed to oppose the world of our thoughts to an external reality, that as soon as we speak of an object to which the forms of our thinking are to be applied, it seems as if we can mean thereby nothing but this external reality. When we call to mind the natural sciences, which occupy so large a portion of the field of science at the present day, we are confirmed in this opinion; on the other hand, when we think of mathematics and jurisprudence we are likely to be shaken. The external reality supplies neither the objects with which the mathematician deals nor the methods by which he deals with them. That which it yields does but give him an occasion to turn his investigations in this or that direction. The true objects of his enquiry are always nothing but the forms which our intuition or our thinking finds in itself or creates, and of which the appearances of the outer world remind us, without ever perfectly corresponding to them. And his business is, in accordance with laws of reasoning, which at any rate are not derived from any external experience, to develop the countless necessary conclusions which follow from the various possible combinations of these forms. Nor is this development speedily achieved: these con-

sequences do not unfold themselves in such a way that we need but to look on and watch: on the contrary logic has at all times turned to mathematics (for the two are coeval) for examples of delicate profound and fruitful methods of enquiry.

Jurisprudence certainly owes the occasion of its origin to the circumstances of the actual world in which man with his needs and claims is placed; but it tries to shape this world and our relations to it by ordinances, which, though as against nature they are products of our free choice, are yet the necessary consequences of ideas of right and justice, consequences of a truth that ought to be, which has its home nowhere but in our own minds. And so logical acumen is just as constantly employed here also in setting forth ever more precisely and irrefragably the connexion of the several conclusions already drawn both with one another and with the highest principles from which they flow.

Thus both these branches of science show that logic need not go to the external reality to find objects for its application,—that it finds fully work enough in investigating the connexion of that which is possible in thought and necessary in thought,—that finally the inner world of our conceptions is wide enough to contain unknown regions, still to be discovered by means of systematic enquiry.

153. Keeping to this line of thought we may now turn to the natural sciences. Even the external world which we assume is after all an object of our enquiry only so far as (in some way or other which does not here concern us) it has become a world of conceptions in us; we survey, dissect, and investigate not that invisible something which we suppose to lie outside us, but the visible picture of it that is formed in our consciousness. We may believe that we are compelled, as the result of prolonged labour, to accept certain connexions according to law between the unknown parts of this unknown external something; but

all these assertions (whatever they may be) are after all grounded solely upon the relations which prevail either persistently or in succession between the contents of our thoughts. Whatever may be the causes which produce this succession, the laws by which it is regulated can only be known by itself, i. e. by the order in which certain thoughts follow certain others in our minds, by the constant union of some thoughts, and the impossibility of uniting others. It is enough then even for the treatment of the external world to regard it in the first instance as a world of thought set up somehow or other in us; whether the appearances which surround us correspond to a real world of external things, or whether they be products of a creative faculty of imagination in us, guided by unknown impulses, the discovery of the connexion between them will always necessitate the same methods of enquiry.

I wish the reader to bear in mind what I have said as we pass to applied logic. My purpose in saying it here is only to indicate the position taken up in the following enquiries: in the course of these enquiries we do no violence to the ordinary way of thinking; let the reader while he reads these chapters conceive of the efforts of thought as directed to a real external world; only when he finds no notice yet taken of the relation of this world to our thought, I hope he will find a justification of this course in these few prefatory remarks, and be content to wait till the third part of my treatise for an enquiry into the significance of the issue which is here put aside.

CHAPTER I.

The forms of Definition.

154. INNER states, sensations and ideas, feelings and impulses, cannot be conveyed like material things, which may be separated from their original possessor and passed on as they are from hand to hand. We can communicate them only by subjecting our neighbour to conditions under which he will be compelled to experience them or to beget them anew in himself.

If we had to communicate for the first time something yet unknown, which was too simple to be created by thinking, or too complex to be exhausted by it, our only resource would be to produce the *external* conditions of perception. If our neighbour had never seen light, or heard sounds, or felt bodily pain, our only course would be to put his eye within reach of a source of light, to bring waves of sound to act upon his ear, and by the application of a stimulus to his body to let him experience that feeling of pain with which we ourselves had made acquaintance in precisely the same way. If we wish to enable him to recognise a person whom he as yet does not know, the description of the countless little marks which distinguish that person from others will never make sure, but by pointing with the finger we can show him precisely whom we mean. We need do no more than thus barely mention the fact that wherever it is applicable this direct reference

to the object itself or to some likeness of it is always useful. But in view of the questions which here concern us we further presuppose two things,—first a large stock of past experiences common to the persons who are to communicate with each other, and secondly a language intelligible to both parties, to the several words of which each attaches (to a large extent at least) the same ideas. Then by a series of spoken words we call to our neighbour's recollection the ideas conjoined with them in that order which is for him the *internal* condition of his creating or experiencing in his own consciousness that which we wish to communicate.

155. This form of communication also includes much else that our logical enquiry can only take note of by the way. Both poetry and eloquence aim by this method at something more than imparting ideas: they count upon the attachment to the images thus called up of feelings of pleasure and pain, of approval and disapproval, of exaltation and aversion. The effects which they thus produce are powerful but uncertain. Different minds are indeed pretty uniformly organised for the mere apprehension of matters of fact, and their general habits of perception do not change; but in estimating the degrees of emotion which we annex to what we perceive we must allow not only for original differences of temperament, but also for the changefulness of the mood of the moment, which depends upon what we have just gone through. Thus different persons are very differently receptive even of actual facts; still less can we hope by the imperfect recollection of such facts, which is all that speech can rouse, to create in others precisely the same emotion which they produced in ourselves. How much may be done by skilful guidance of the train of ideas and by well-measured expressions to lessen the uncertainty of the result is a question for the art of poetry and rhetoric. Our own problem is narrower and is limited to the communication of that which has been

already refined from a state in which we are acted upon into an idea which we apprehend,—i. e. of thoughts, not of feelings and moods.

156. The certainty even of this kind of communication seems to be imperilled by the fact that after all the same words do not always have the same meaning for the speaker and the hearer. It must be allowed that, apart from subsequent confusion of originally different roots, there are in every language many words which denote several very different things,—in consequence no doubt of a resemblance which these things bear to one another, but still of a resemblance which is not always so obvious now to him who uses the traditional words as it was to the first inventor of these metaphorical expressions. And even when a word denotes the same thing for all, that does not ensure that all have the same conception of the thing denoted. The special circumstances under which each individual became acquainted with the thing, the peculiar point of view from which he first regarded it, the connexion in which he found it and from which he had to detach it, give a peculiar colouring to his picture of it, and dispose him to other conclusions than those anticipated by the speaker when he named the common word, hoping thereby to give some particular turn to the course of his hearer's thoughts. It is impossible to deny these facts, dangerous to disregard them altogether, yet foolish to press them too far: the intercourse of daily life sufficiently proves to how large an extent speech enables us in spite of them perfectly to understand each other's thoughts about the most various matters. There will certainly remain ideas which it is hard to communicate with precision; but were there no such difficulties there would be no good in seeking rules for helping us by the appropriate use of unequivocal words to remove the ambiguity of others and to fix their meaning so that all who wish to converse may use them in the same sense. It must be left to the unfettered acumen of the speaker to

determine what words may be accepted as precise enough to explain other words; but however far we may feel constrained to go back along this line and to remove all ambiguity from the instruments of communication which we wish to use before we use them, there will still be only two possible ways for us, abstraction and construction.

157. We explain a conception, which we will call M, by *abstraction*, when we first refer to a number of known instances, in each of which M forms a part of the notion, and then bid the hearer separate from these instances that which does not belong to the conception M which we wish to communicate. This is the way in which all our general conceptions[1] and general ideas[2] were originally formed; in the case of a general idea that which was common to a number of impressions comes of itself to stand out as the object of a new separate idea; in the case of a general conception this process is consciously directed by attention and reflexion. And when we are at a loss we all come back to this same way. The man of no logical training does so when to the question what he understands by M he replies, in the fashion which the Platonic Socrates so often complains of, only by giving examples which contain M, leaving to his questioner the trouble of separating the common element which he wants to get at from that which is foreign to it. But the logically trained thinker also proceeds really in the same way: however carefully he may choose his terms so as to express the universal itself without any reference to particular instances, yet this expression is only obtained by a tacit comparison of a number of cases. It is only by such a comparison that we learn what marks of M must be precisely fixed in order that the expression may exclude all that is foreign to M, what other marks must be left undetermined in order to include in M everything that is properly an instance of it. And lastly, only by the fact that instances are to be found are we convinced

1 ['Begriffe.'] [2] ['Vorstellungen.']

that this *M*, which we are taking the trouble to determine, is capable of determination, that it represents a problem which has an intelligible solution, not a mere tissue of incompatible elements whose union may be demanded in words but cannot be really carried out.

158. It is thus useful to follow this method of abstraction in every case, and even when we may have arrived at a determinate conception in some other way, at any rate to confirm it by a supplementary reference to instances. Wherever our aim is to fix some very simple conception which underlies a whole group of kindred ideas, it is the only method possible. Such a conception can only be pointed out by taking away from known instances of it all that does not belong to it; we can never put it together out of its component parts, for it has none. The labour expended upon this impossible aim always ends in a vicious circle, since among the materials that are to be used in the construction the very thing that was to be constructed is taken for granted, whole and entire, however much it may be concealed under strange expressions. Thus, for example, in our idea of *becoming* the two ideas of being and not-being are no doubt united as two connected points of relation; but if we should try to characterise becoming as the unity of the two we should not attain our object. In the first place we should be bound to fix the precise sense to be here assigned to the expression 'unity' which in itself is very ambiguous. It cannot mean the mere co-existence in the same consciousness of the two ideas of being and not-being, for obviously becoming is the content of a relation that exists between the contents of these two ideas. But if we try to unite being and not-being as predicates applicable at the same time and in the same manner to one and the same thing, we do not arrive at becoming, but simply find ourselves confronted by the impossibility of actually executing in thought a task which involves such a contradiction. Suppose then that we separate again the being and the not-

being of this thing and say that the one predicate is applicable to it when the other is not:—even by this change we do not get hold of becoming; it falls between the two moments of time and is to be found in neither. We shall have therefore to bring them together once more: but as long as they are separate from one another becoming will lie outside of them, we can only get hold of it when we look for it neither in being nor in not-being, nor in a *passive* unity of the two, but in the transition from one to the other. But in this idea of transition, or in any idea however it be expressed that we like to substitute for it, we shall recognise (only under another title) what is essentially our idea of becoming. *This* relation therefore between being and not-being, as it is altogether *sui generis*, cannot be conceived by means of anything but itself,—is only to be got by abstraction from the instances in which it forms a part of the thought, not to be created by the putting together of ideas which as yet do not contain it. Precisely the same considerations hold with respect to the equally simple conceptions of being, acting, thinking, affirming, denying; and the geometry of Euclid follows precisely the same method in determining the surface as the limit of the space occupied by a body, the line as the limit of the surface, the point as the limit of the line,—in each case teaching the learner to get the simpler conception, which is harder to grasp, by abstracting what does not belong to it from the more complex conception which lies nearer to sense or which has just been determined.

159. The opposite method would fully deserve the name of *construction* only if it enabled us completely to put together the idea to be conveyed out of a definite number of unequivocal parts by a series of acts of thought which we were required in unambiguous language to execute upon those parts. Almost the only conceptions that really admit of this treatment are the mathematical conceptions and some others that arise out of the applications of mathe-

matics,—conceptions which as creations of our thought contain only what our thought has combined in them. They admit of it because the several ideas which make up the whole conception can be completely enumerated, and because not only each of these ideas but each of the ways in which they are to be joined together is such that we can state the characteristic quantity by which it is distinguishable from others of its kind, as well as the special quality which distinguishes it from those of another kind. Here then nothing remains indeterminate that should be determined; he who follows the directions given must see the picture he is desired to form rise before his mind's eye with just that degree of individuality or generality which the speaker wished to give it.

If on the other hand we wish to convey a notion of some really existing thing we are met by well-known difficulties. Our mental picture of a real thing is not made up of a limited number of points of relation which are to be brought into combinations also limited in number, but is compounded of a countless number of ideas. And of these component ideas those that belong to different senses cannot be compared with one another, while even those of the same sense can only be designated by general names, and scarcely admit of precise measurement. And lastly it is beyond our power to make a complete survey of the combinations of all these elements, nay we cannot perceive them at all except so far as they consist of an external arrangement in Space and Time, and even then we cannot find any comprehensive expression for them in our ignorance of any pervading law of their formation.

In the presence of this fulness of detail construction shrinks into *description*. In describing we try, if we understand our business, first to fix the main outlines of the whole idea, whether this be done by a simple construction, or by taking as illustrations similar things already known and proceeding by alteration and transposition, by the

removal of some features and the addition of others, to elicit from them the leading lines of the picture we wish to convey. Then we fill in the mass of details, never completely, for they are usually inexhaustible, but skilfully selecting those by the mention of which we may hope that the hearer's attention will be at once stimulated to supply from his own memory those that are not mentioned. We need but remind the reader of the wonderful effects which the poet produces in this manner, bringing a whole picture before us with a touch; though the uncertainty of the result is equally manifest. The way in which each man supplies what is not mentioned varies according to his nature: were it possible to bring to view in detail the different pictures which the same description calls up in different hearers, their variations would show what an inadequate basis a description must be for the support of definite conclusions. For scientific purposes therefore description needs a regulation of its method, and this it finds in the rules of definition.

160. For the *definition* of a conception M it is usual to require a statement of the next higher generic conception G (the *genus proximum*), and of the characteristic mark d (the *differentia specifica*) by which M is distinguished from other kinds of G. By requiring the generic conception G we set bounds to the arbitrary and capricious course of description. In describing you were free to begin at any point whatever, and then gradually to add the remaining points in any line that you pleased, so long as you could be sure of producing in the end a clear picture of what you meant. But even in a description you would not attain your end without the employment of many general conceptions. Now instead of an arbitrary choice of these, the rules of definition require you to start from that universal conception in which the largest part of the constructive work before you lies completed and ready to hand, and which, being denoted in speech by an unequivocal name,

may be assumed to be familiar to every mind, fitted to serve as the outline for the filling in of the details by which the intended picture is completed.

If we are told that a creature we have never yet seen is a bird, this general conception gives us at once a clear picture of a number of members united in a characteristic manner, and at the same time of the peculiar kind of locomotion and vital action to which they are instrumental. The further special characteristics are easily added to this outline, for it indicates of itself the places to which they severally belong. We should never get such a clear idea of the unknown creature if we had to put it together out of its primary components. It would be an endless task to enumerate all the variously-coloured spots on its body with their position and the extent to which they may be displaced, so as to give a notion even of what it looks like. Still more endless would it be to add to this the peculiarities of life and habit, which all belong at any rate to our idea of the animal in question if not strictly to our mental picture of it.

We see then the value of the abbreviation effected by starting from a general conception that can be assumed as known: we understand also that we must choose for starting-point not merely any higher universal, but expressly the *genus proximum*, which in its characteristics and in the mode of their combination comes closest to the conception to be defined, and so clearly describes the point at which and the manner in which we are to add each of the last characteristics by which the conception is finally determined. By starting from a higher universal than this we should not only lengthen again the rest of our task, which definition was intended to shorten, but we should run a risk of failure. For we should then have to add a whole series of further characteristics in order to exclude everything foreign in the long descent from that less determinate universal to the particular species in question: and each new characteristic

would open a new source of error; for it is hardly possible to determine quite precisely the mode and manner in which each is to be added to those that have preceded it without appealing to a picture which it may be assumed that each man already has in his mind. The notion of that *genus proximum* therefore would not by this method be produced afresh with that definiteness and certainty with which it could be recalled to the memory at once by the mention of its name, and which it must have if it is to serve as an outline for the filling in of the final characteristics of the conception which we desire to convey. All that we could get by this method would be more or less of a riddle. For when we propound a riddle what we do is this,—we tell our hearers without more ado to attach to a very indefinite universal (a mere something that may be anything) predicates that can be united only in one very definite subject, leaving it to his ingenuity to find this subject or in the first instance the *genus proximum* which admits of their union.

161. As yet we have spoken of the definition as a methodical description. If it is to retain this character it would have with regard to M to state completely the modified forms $p^1\ q^1\ r^1$ assumed in the case of M by $P\ Q\ R$ the general predicates of the genus G. Instead of all these characteristics the usual rule for definition requires us to set down only one characteristic d, the specific difference, by which M is distinguished from all other species of the genus G. Definition thus has a more limited and therefore a more practicable aim than description: instead of setting forth positively the whole content of M it has only to state the mark by which M may be separated from all that is not M. This is the origin of the terms *definitio* and *ὁρισμός*, both of which imply only the marking off of one thing from another. And in fact the general aim of definition must be thus limited. As thought advances we feel no doubt the need not only to distinguish, but to know completely what we have distinguished; then we make further demands

upon definition; then we refuse to admit as a specific difference anything but one of those characteristics that really make a species, i.e. one whose occurrence decisively modifies the forms assumed in M (the thing to be defined) by all the other characteristics of the genus G which are not mentioned in the definition. These heavy demands however can be completely satisfied only at the conclusion of an enquiry which has made us perfectly acquainted with the nature of M, and which thus enables us to solve the problem which remains, of fixing a final and classical expression for that nature.

But besides this there are other no less pressing problems. We may have to begin a speculative enquiry, which has to find a number of yet unknown propositions that are true of M; or in a practical matter we may have to determine what is the proper consequence of a given situation M: in either case it is of the utmost importance that this M, to which the propositions we are going to assert or the decision we are going to arrive at must apply, should be marked off by precise and easily traceable boundaries,—nay at first this is the only thing that is of importance. For this purpose any characteristic d will suffice, even the most insignificant, provided only that it be really an exclusive mark of M. In the first case, that of a speculative enquiry, the further course of the enquiry itself will either reveal the reason which connects the validity of a series of propositions with the presence of this obscure characteristic d, or will show that they are valid over a wider or narrower field than this, so that d is not the proper characteristic of their subject. In the other case, that of a practical matter, the exact meaning of a legal situation to which a law is to apply must be completely considered beforehand while the question is still *de lege ferenda*; but he who has to carry out the *lex lata* rightly demands that this previous consideration shall have given the law the form of a definition which distinguishes, not by the most profound but by the most obvious mark,

the cases to which a decision shall apply from those to which it shall not. These are problems which applied logic cannot decline, and we overlook them when we think too disparagingly of this traditional form of definition. We misunderstand the sound sense of many such definitions in practical philosophy and jurisprudence when instead of the marks of *M*, which they intend to give and do give completely, we see in them nothing but an inadequate statement of the whole nature of *M*, which it is not their purpose to give at all.

162. It will be convenient to notice in this context the distinction which is commonly drawn, but not always in the same sense, between *nominal* and *real* definitions. We may utter a name or replace it by another; but we can never define anything but its meaning, i.e. our idea of that which it is intended to signify: the thing itself again is not in our mind, but only the picture we have formed of it. These two kinds of definition therefore seem to be identical; and they are in fact identical for everything that exists only in our minds, and whose whole nature therefore is exhausted by our idea of it. There is no real definition of a geometrical figure that can be distinguished from its nominal definition; any correct definition that we give of it expresses at once the whole nature of the thing in question, and the whole meaning of the name.

In other cases however the distinction between these two modes of definition is one that it is worth while to make. If we call the soul the subject of consciousness, of thinking, feeling, and willing, this may be appropriately called a nominal definition; it specifies a condition which a real thing must satisfy if it is to be entitled to the name of a soul. But who or what this thing is whose peculiar nature enables it to satisfy this condition, is still quite an open question; we have not fixed the real definition of the soul till we have got a theory which proves either that only a supersensuous and indivisible being, or that only a con-

nected system of material elements can be the vehicle of consciousness and its various manifestations. It was a nominal definition of beauty that Kant gave when he said that it is to be found not in the conformity of the beautiful object with some conception, not in its capacity to satisfy a desire in us, but in the fact that it pleases directly and without reference to any interest. The real definition of beauty would have to point out the precise relations between various things or components which enable every object in which they occur to produce this pleasing effect. And so we may say in general terms, when experience shows us a group of characteristics $p\ q\ r$ often occurring and continuing together, or when in the course of our investigations we light upon a coincidence which induces us to put them together and to regard the group as a subject for further enquiry, we proceed in the first instance to form for the group a conception M, of which a nominal definition can always be given, because it has only to set forth the predicates which led us to invent the name, or the effects which we expect from the thing to which the name is applied. But a real definition cannot always be given: for there is no assurance that we have not combined in M characteristics whose union we thought ourselves justified for some reason or other in assuming or desiring, when there is in fact nothing to be found in which they really are or can be united. It is a common error to mistake this mere indication of a problem we should like to solve for the solution itself; and on this account the distinction between these two kinds of definition is useful as a warning.

163. We have to beware of three faults which vitiate a definition.

In the first place its assertion $M = Z$ must be no tautology; but it becomes a tautology whenever M itself is explicitly or implicitly assumed among the ideas combined in Z by which M is to be explained. This fault (called *circulus in definiendo*) is often committed through carelessness which

no rules can prevent; but we are almost of necessity driven to it whenever we try to give a formal definition of some simple thing which does not fall under any more general conception.

In the second place a definition, since it has to fix a conception, must be a universal proposition, true of everything which falls under the conception. Now if every $M=Z$, it follows by contraposition, that no M is not-Z: if then further reflexion or fresh experience teaches us that after all there are some M which are not-Z, we know that the definition $M=Z$ was too *narrow* (*definiendo angustior*) and was not, as it ought to have been, true of every M.

Lastly a definition must be convertible: if every $M=Z$, it must also be true that every $Z=M$: whenever therefore further reflexion or fresh experience shows that some Z are not M, we know that the definition $M=Z$ was too *wide* (*definiendo latior*), and included some non-M which it ought to have excluded.

To point out how to avoid these faults would be more useful than thus merely to name them; all we can do in that way however is to indicate their usual source, viz. the limited range of our observation, which as a rule opens to each individual only one and the same fragment of the entire field covered by a conception, and further the one-sidedness into which our thinking is apt to lapse if it does not constantly receive fresh stimulus from without. In the temperate zone the way in which plants awake in summer and sleep in winter makes a strong impression upon our feelings; animal life, with its continuous activity, seems to offer a complete contrast. Now we certainly should not base upon this a scientific distinction between animal and plant; yet countless comparisons, employed by poet and orator, show that we are accustomed to consider this yearly alternation as the essential characteristic of the plant. But a definition which expressed this would

be at once too narrow and too wide: it would exclude tropical plants whose life is an uninterrupted growth, and would include hibernating animals, which in this climate easily escape our attention, directed as that is mainly to the domestic animals. It may easily happen that one who wishes to establish on a new basis the rights and duties, both political and social, of all the members of the state, thinks only of the male world to which the conduct of these transactions is usually confined, and then his proposals will be too wide, in as much as he demands for all what he intends for men only, or too narrow, in as much as he expressly enacts for men only what must obviously apply to all. From this we may draw a lesson of universal application: we should never attempt to treat a problem off-hand, when it is possible to extend the limits of our own experience by converse with others or by taking count of views which are already recorded in the literature of the subject. Learning is not in itself inventive, but like any other training and discipline, it makes us more secure against extreme errors than if we proceed by the mere light of nature.

164. We further require in a definition elegance and brevity, which I will illustrate by a simple instance. If we define a circle as a curved line all the points of which are equidistant from its centre, we first of all make an actual mistake in giving too wide a definition. For if on the surface of a sphere we draw a serpentine line which crosses and recrosses a great circle of the sphere making equal curves on either side, all the points of this line are equidistant from the centre of the sphere.

If further the line, in returning to its origin in the great circle, describes an uneven number of these double curves, it will consist of an infinite number of pairs of points, forming the opposite extremities of so many diameters of the sphere. The centre of the sphere therefore bisects the rectilinear distance between the two points of each pair; and

so, in every sense which can here be given to the word, it would also be the centre of the sum of all these pairs, i.e. of this line, which nevertheless would not be a circle. We ought therefore to have said that a circle is a curved line *in one plane* which fulfils the above condition.

But elegance further demands that a definition shall not contain more ideas than are indispensable for the complete determination of the given conception. So we may be called upon to speak not of a curved line but of a line simply: if a line fulfils the annexed condition it follows without more ado that it cannot be straight. The condition itself however is not correctly expressed. A definition should not employ among its instruments of explanation ideas which are themselves unintelligible without the conception to be defined. In this case the idea of the centre is certainly such an idea. If we had not yet got the idea of a circle (and in fact there is nothing in this case at least to suggest this idea to us, after we have omitted the characteristic of curvature from our definition) we could at first think of the centre of a line only as the point of bisection, and we should not discover our error till we attempted to construct a circle on that understanding. Instead therefore of this sense of the term centre which common usage suggests, and which compelled us to be so painfully discursive just now in speaking of our serpentine line, the definition requires the precise statement in general terms of the meaning which the word is to bear for all figures whatsoever. This statement can easily be given, but I may omit it, as it follows therefrom that *if* there be a point in a plane which is equidistant from all the points of a line in that plane, that point is the centre of the line. But if we now introduce this definition of centre into our definition of a circle, the statement of the further condition under which the line in one plane becomes a circle comes to be a mere tautology, and the meaning of the whole definition is evidently nothing more than that a circle is a line in one plane for which

there is a point in the same plane from which all its points are equidistant. The definition is substantially correct; yet fault may be found with its form. For now after omitting the term centre we remember that it was only the presence of that term that forced us to look for the equidistant point in the same plane. Not this actual centre only, but any point in an axis drawn through it at right angles to the plane of the line fulfils the condition of being equidistant from all points of the line. It is enough therefore to say that a circle is a line in one plane such that a point may be found from which all its points are equidistant. It is needless to mention that there are several such points and to say where they lie: the attempt to construct the line according to this direction will at once teach us both. But once more even in this form the definition is not quite all that can be desired. It does indeed say that all the points of a circle are equidistant from one and the same point, but it does not formally state whether or no all points that are equidistant from this point are points in the circle. They are so in fact provided they lie in the same plane, and thus in order to express this along with the rest we may finally say that a circle is a line which contains all the points in one plane which are equidistant from any point.

165. Different opinions may be entertained as to the requirements of definition which I have just illustrated by the example of the circle. Every one will allow that it is a serious fault to employ ideas which (like centre in this case) though a meaning may be given them apart from the conception to be defined, yet are not fully intelligible without it, except perhaps in the context of a scientific treatise. But it may be thought that the addition of superfluous characteristics is unobjectionable, since it makes the definition easier to understand without impairing its correctness. Nevertheless it should be avoided. For the addition of some characteristic *z* that might be dispensed with, is apt, as we are not told that we might dispense with it, to make

us think that it is inserted in order to distinguish the *M* we are defining from a non-*M* to which everything in the definition is applicable excepting only *z*. If we say a circle is a curved line in one plane such that there is a point from which all its points are equidistant, the form of the statement suggests that there are also straight lines which satisfy that condition. It matters little in so simple a case as this; but in more complex cases serious disadvantage may be the result of this apparently harmless addition of superfluous matter. At the least it hampers us in the drawing of conclusions, which after all was our sole purpose in laying down the definition. It may happen, for instance, that it has been quite clearly established, perhaps in some indirect way, that *Q* has the whole sum of predicates that are sufficient according to the correct definition for the subsumption of *Q* under *M*, but that it is difficult or impossible to prove directly that *Q* also has the predicate *z* which is superfluously added in the definition actually given: there will then be a quite useless hesitation about bringing *Q* under *M* and actually drawing the conclusion which that would justify. And so we may say generally that it is right to demand that a definition shall contain only those terms that are indispensable for the specification of the object, but shall exclude all merely descriptive elements: if it does not enable us very readily to form a picture of the thing, this will be atoned for by the certainty of the conclusions we can draw from it.

166. Hitherto we have been considering the usual form of definition by the proximate genus and the specific difference as the only valid form. But the untrained intellect is wont, to the annoyance of the logicians, to use another mode of definition, and to say, for instance, in its familiar uncouth way, sickness is when something pains me. Such a phrase certainly needs to be amended, yet not exactly in the way which logicians rather intolerantly require, but rather in the way in which physical science

actually defines many of its conceptions. The ordinary form is properly adapted only for defining the meaning of a substantive: when we have to do with adjectives and verbs it is not only shorter but more correct to give them their proper place in the grammatical structure of the definition, and to let them bear plain reference to their subject, seeing that it is only as expressing states or properties of a subject that they have any meaning. It is quite right therefore to define adjectives like sick or elastic by such propositions as 'a living organism is *sick* when its functions depart from a certain course;' 'a body which on the cessation of external constraint resumes its original shape is *elastic.*' And in defining the meanings of the verbs *to live* and *to sin* it would be quite proper to name first the subjects to which they can be applied, an organic body and a spirit that is conscious and wills, and then the conditions under which they are to be predicated of *these* subjects. It is absolutely useless to begin by throwing all these ideas into the substantive form and ranking them under the head of states or properties or modes of action: that they are to be so ranked is at once apparent if we leave them their adjectival or verbal form and give them their proper place in the sentence. The usual mode of definition on the contrary has the disadvantage of making us far too apt to separate from its subject and treat as independent what is nothing but a state or property of something else. When we have once framed the substantives sickness, sin, freedom, it is hard to keep quite clear of the strange mythology which speaks as if these terms stood for things with a being of their own, and traces their development, without ever seriously coming back in the course of its enquiry to their real subjects, though it is only as properties, states, or activities of these that they exist, and though their apparent development is every moment bound up with the real development of these subjects.

167. Under the head of conceptions to be defined we

have hitherto considered only comparatively simple ones, conceptions of figures, things, properties, and easily intelligible relations: but among the words used in speech, every one of which may under certain circumstances call for a definition, we often find very complex relations between a great variety of points of attachment comprehended in one simple expression. No one who was not hide-bound by prejudice would require that the explanation of such conceptions should take the regular form of a simple definition; and to find special names for all the other very various methods which may be employed would be nothing but useless pedantry. The universal principle of applied logic is simply that all ways are allowable which lead to the goal; it hopes for no more than to remove our doubts as to which way is passable right up to the end, and which not, by pointing out that which has long ago been tested: it never forbids our seeking new ways to satisfy new needs. It is always allowable therefore to begin with a preliminary description, with comparisons and analogies, with discussions of any kind, in order to familiarise the hearers with the meaning of the subsidiary ideas we wish to employ and the peculiar combinations we wish to establish among them, and having thus prepared the way to proceed to set forth what we wished to explain in a formula which is brief and intelligible, though it presupposes what has gone before and cannot be separated from it.

This reminds us however of another twofold division of all definitions. We may characterise M by the aggregate of marks displayed by the conception when it is present to our minds in its completeness: this kind of definition, which we illustrated just now in the case of the circle, may be called *descriptive* definition: we have recourse to it mainly in the case of actual things which we only know from the outside and whose definition therefore is in fact nothing but a methodical description. But we can also fix

M by pointing out a way in which, not by the mere addition of other ideas, but by freely using and manipulating them at will, *this* idea can be produced with certainty. This I would call *genetic* definition, understanding thereby (and this I wish particularly to emphasise) not a statement of the process by which the content of the conception *M* is actually found, but only an indication of the way in which the *mental picture* of this content *M* may or must be formed. 'Let a straight line revolve in one plane about one of its extremities, and combine the successive positions of the other extremity:'—that is a genetic definition of a circle. The circle as such is not made at all: but supposing a particular circle such as we draw to have been already made in some way or other, we may certainly form a mental picture of it in the way indicated by this definition. But we may form that mental picture equally well by supposing the length of the two axes of an ellipse to alter till both are equal to *r*; or by supposing a cone to be intersected by a plane at right angles to its axis. And thus an idea, whose content has in itself no genesis, may admit not only of one, but of so many genetic definitions as there are ways of forming the idea of this content by the manipulation of other ideas. Among these genetic definitions then, using the term in a somewhat extended sense, we may include the above-mentioned miscellaneous methods: they try by indirect means to make us form a mental picture of *M*, when it is impossible or inconvenient to say directly what *M* is.

168. Strictly speaking, whenever we undertake to define a conception *M*, our aim is to give it a higher degree of definiteness than it has yet. But in fact the problem usually narrows itself to the transformation of a *clear* idea (*clara perceptio*) which we already have of *M*, into a *distinct* one (*distincta*), or of a *mere mental picture*, which does but comprehend *M* in a loose general way as a connected whole made up of parts which are familiar, into a real *conception* of

M. These two expressions may be regarded as equivalent. For according to old established usage we are justified in saying we have a clear idea of anything when we think of it as one, and as a connected whole, and lastly as distinguished from others with precision enough to avoid confusion; but it does not become distinct till to this is added the general law which regulates the connexion of the parts, and further the characteristics which it has in common with other species of a certain genus, and lastly those particular characteristics which distinguish it from all the other species of its own genus. In treating of Pure Logic we identified this increase in definiteness with the transition (in technical language) from an idea or mental picture to the conception or actual comprehension of a thing.

But now there are cases in which our idea of an *M* which is to be defined is far from possessing the clearness here supposed: names are handed down to us which have become part of our language though their meaning has never been precisely fixed. Thus we speak of virtue and sin, of good and the highest good, of appearance and reality, with a full conviction that we mean something very definite by these names, and ready to draw important inferences from them in reference to that to which we apply them. But at last the difficulties in which we entangle ourselves convince us that strictly speaking we did not know precisely what we meant, that we had not completely fixed the conditions which must be satisfied in order to justify the application of these names, that we had in short trusted to hazy ideas, the clearing up of which is of the very first importance. This we try to effect in a very simple way. If we were entirely ignorant of the meaning which *M* was intended to bear, we should have no means of finding it out; but also it would never have occurred to us to apply this name had not some part of its meaning (say *a*) been fixed beyond a doubt—that very

part namely which now impels us to use the term the rest of whose meaning is still hazy. This a we first take tentatively as a complete definition of M, and consider whether a corresponds to what we *mean* by M. It is a matter of common experience that in cases where we are not in a position to express the meaning of M in positive terms we may yet see whether an idea a that is offered as a definition of it is adequate or not. Thus when we are trying in vain to recollect a name we can yet pronounce with perfect certainty that a suggested name is not the right one; and further any resemblance it may have to the right one makes an impression on us, and sometimes reminds us at once of what we want,—at any rate it helps to make plain the other points in which the right name differs from the suggested one. We are in the same case here: a is not utterly wrong and incapable of comparison with M: the comparison of the two therefore does not lead to the bare negation of their identity, but puts us on the track of a supplementary b which must be added to a, or an alteration b which must be effected in a in order to make it answer exactly to M. Now putting M down as equal to $a + b$ we make a second attempt and repeat the same course of comparing and supplementing by fresh terms c and d, till at last we get a definition $M = a + b + c + d$ which in its expanded sum of characteristics exactly coincides with what we meant by M. In this very simple process of thought rather than in a strictly inductive method lay the art which the Platonic Socrates used ages ago to clear up hazy conceptions.

CHAPTER II.

Of the limitation of Conceptions.

169. In the course of an investigation we may be led by a definite purpose to trace a group of characteristics *i k l* through all the otherwise different objects in which it occurs, and to ask what influence is exercised by its presence upon the rest of their characteristics. The result of this comparison then will itself teach us whether the other characteristics which each of these subjects has in virtue of the genus to which it belongs are modified by the presence of *i k l* in any remarkable and particularly in any constant manner. If this is the case we often form out of *i k l* and out of the idea of a more or less precisely determined subject a new generic conception *M*, treating all the ideas in which *i k l* occurs as species of *M*. But whenever this is not the case (and not seldom too when it is) we content ourselves with treating the presence of *i k l* as one of the countless variable conditions, which affect other ideas so far as to necessitate certain alterations in them, but do not themselves form a generic conception under which the several instances in which they occur could be arranged as species. Now a living language is believed by those who use it to have already sufficiently distinguished in the coinage of its words the two kinds of cases in which these two methods are severally appropriate. Of course they will allow that enquiry, as it goes deeper and deeper, will discover many a new group

of characteristics *i k l* having such a decisive influence upon the whole bearings of every conception that contains it as to make it worth while to erect this group into a separate generic conception *M* and to mark it by a name: and language is in fact constantly enriching itself by new names for the ideas thus newly discovered. But, on the other hand, they will also assert that none of the conceptions already found and fixed by the creation of a name are unworthy of this distinction: each, they insist, really means something coherent, which is thus justly cut off, as a whole with well marked boundaries, from all other similarly coherent ideas.

170. These conceptions which our inherited language supplies are the tools with which our thought must work—and that not merely because we have no means of communication except the words which have been invented to express them: in this store of words is treasured up the concentrated result of the thought which the human mind has from the earliest times bestowed upon the world to which it has access, and we may suppose that the same impulses which led it to fix its conceptions in this form would also in the first instance assert themselves in us were we to go through the same labour.

But that these impulses, however natural they may be to man, yet leave room for doubt is shown by the divergence that constantly occurs in the application of the conceptions thus formed. When the question arises whether some predicate *P* is to be affirmed or denied of a subject *S*, one maintains that *S* is a kind of *M* and therefore is a *P*; another objects that *S* is no *M* and therefore no *P*; a third allows that *S* is indeed no *M*, but an *N*, but declares that this does not matter, and that what is true of *M* holds good of *N* also, while a fourth insists that the difference between *M* and *N* establishes a difference between the two in respect of *P*.

The divergence that here shows itself culminates in two

opposite tendencies, dominating the whole of our thought. The one is a tendency to exaggerate every difference that presents itself into absolute difference, and with the familiar formula 'this is something quite different' to resist all argument from one case a to another case b which resembles a but is not exactly like it: this tendency becomes in life and in science the spirit of the pedant and the philistine. The other is a tendency to ignore the fact that a difference which is not absolute difference may yet have a qualified value, and with the barren phrase 'all is one at bottom' to obliterate all the fixed boundaries which define the province of each conception, thereby destroying the only grounds upon which certain predicates are attached to certain subjects and to no others: this becomes in thought and action the principle of a no less ruinous libertinism. A glance at the momentous consequences of these confusions makes us alive to the necessity of clearly understanding what reasons there are to justify us in dividing the whole extent of the intelligible world into definite conceptions, where the boundaries of their several provinces are to be drawn, and what value is to be assigned to this demarcation.

171. We are led to very various issues by the attempt to answer these questions even where they are easiest and least pressing, viz. in regard to the simple contents of sensuous impressions. We have a right to assume absolute difference between simple sensations A B C when we cannot imagine any intermediate steps by which the peculiarity of one could gradually pass over into that of another, and when further we cannot think of any mixture of two of them which would give a new simple sensation, and when lastly there are no degrees of contrast between them such as would enable us to estimate the difference between A and B as greater or less than that between A and C or between B and C. We find these relations, or rather this lack of any assignable relation, between A B and C if A stand for

colour, *B* for sound, and *C* for smell. We may keep the old name and call them *disparate* or incomparable.

This conclusion will not be affected by various secondary considerations which may be urged. It may be pointed out, for instance, that all three exist only as states of our consciousness. To this we reply that they all are indeed sensations, and may be called, according to the usage of logic, species of sensation; but that the conception of sensation in general cannot serve here as a generic conception in the sense of supplying a law for formation. When we think of the shape of an obtuse-angled triangle as subordinated to the general conception of a triangle, we have in the latter a constructive formula, whose application has but to be varied within its own limits in order to show us that there are right-angled and acute triangles besides that one species from which we started. But the subsumption of colour under the general idea of sensation (for it is only subsumption that is possible here, not subordination) can never enable us to conclude from this general idea that there are such sensations as sounds and smells besides colours. Although these three then are, to use the ordinary phrase, kinds of sensation, yet within the limits of this universal they remain quite disparate the one from the other.

Again as states, as motions or affections of the soul, these various kinds of sensation may produce certain secondary effects that are comparable with one another, and it is certainly allowable on that account to compare a certain colour a^1 with a certain sound b^1 or a certain smell c^1: but still that which produces these comparable after-effects remains itself quite incomparable. And we must make the same reply to the physicist and the physiologist, when the processes which must take place in the outer world or in our nerves in order to produce the various kinds of sensation are traced back by them to comparable, or perhaps even to closely allied movements of material particles:—

they must conclude not with the curious assertion that there is therefore strictly speaking no qualitative difference between these sensations, but rather with this other assertion which is true, viz. that in spite of the similarity of origin there is not the slightest similarity in the results. There is no room for doubt here, except in so far as the unprejudiced observation of ourselves, which is here the sole criterion, is unable to pronounce decidedly. This is the case with regard to taste and smell. Sourness is undoubtedly common to both; but the other sensations of taste and smell also seem to form a connected group, only that some members of this group are excited only by the agency of liquids, others only by that of gaseous matter. It may be that the sensations of these two senses, which on this account must have different organs, are themselves homogeneous and distinguished only by secondary sensations dependent upon the position, shape and action of their respective organs. But it is not the business of logic to decide this question: all we need do here is to warn the reader when he has a direct perception that two modes of consciousness *are* incomparable, never to allow this to be overborne by sophistic arguments based upon the similarity of their *antecedents* or *consequents*.

172. The other question, not as to our right to separate A and B, but as to our right to join together all that we comprehend under A, calls for a similar remark. For a long time people tried to dazzle the public with the stupid paradox that black and white were no colours because they did not like the prismatic colours depend upon a definite number of undulations of light. The progress made of late in the physiology of vision has completely cut away this ground; but even if this had not been done, no one could have had the right to override language in this fashion. Long before we knew anything about the exciting causes of our sensations, language had invented the name of colour for a group of sensations which by a homogeneous quality

directly perceived and undeniable, viz. by *shining* or whatever else we like to call it, are at once bound together and separated from tones that *ring* or resound and scents that are *smelt*. Granted that the name shining is only appropriate to white and not to black, still the fact that the fundamental quality thus imperfectly designated is shared by both in common with the other colours admits only of a verbal not of a real denial, and the common usage of the term colour so as to include both was therefore completely justified against the unsupported objections of the savants.

In other fields also we find similar instances of the encroachments of scientific theory, not always harmless in their results. Thus chemistry for a long time contributed to the confusion of speech by identifying oxidation and burning. Men assuredly spoke of burning long before they knew of oxygen, and always meant by it a process accompanied by visible light and sensible heat, which permanently altered the constitution of a body: a glowing iron rod therefore was not said to burn, because no lasting alteration was found in it when cooled: but also such a permanent change would not have entitled the process which produced it to the name of burning, in the absence of the sensible development of flame and heat. The notion of burning then by no means coincides with that of oxidation: many substances are oxidized without burning, and on the other hand, when heated antimony is immersed in gaseous chlorine and combines with chlorine, throwing out flames the while, this process is undoubtedly one of burning though not oxidation. Geometers, again, knew ages ago that any system conceived in abstract terms, i. e. arithmetically, provided that not more than three scales be required for the arrangement of its various elements, may be presented to our perceptions by means of spatial constructions. Now there is nothing to prevent a mathematician from conceiving systems based upon any number of scales greater than three, only it is plain that such systems can no longer be envisaged in space,

and that the name 'dimensions' which could be applied to the scales in its ordinary sense of dimensions of space so long as they were only three, can now bear only the more abstract sense which I tried to express by calling them scales. As space therefore means for us nothing but a system that we envisage in this peculiar way which certainly cannot be derived from any considerations of mere number, to continue to speak of a system of four or five dimensions as space is but to make sport of logical distinctions. Let us be on our guard against all such attempts: they are nothing but scientific freaks, which intimidate the popular consciousness by utterly useless paradoxes and make it doubt its well established rights in drawing the boundaries of its conceptions.

173. When we now ask how the several coherent members of one of those disparate kinds of content *A B* and *C* are related to one another, we find that these relations are peculiar and not always of the same kind. No one has yet succeeded in reducing the several kinds of taste to a satisfactory system: but the path which common usage takes in naming them, incomplete though that nomenclature be, seems to me the right path. Certain primary forms are distinguished by names of their own, as sweet μ, sour ν, bitter π, and the others such as sour-sweet $\nu\ \mu$, bitter-sweet $\mu\ \pi$, are regarded as compounds of those well-marked primary tastes. Our imagination could never have lit upon this mode of naming them had it not been guided thereto by the direct impressions of sense, for we cannot make differences unless they are already present actually or potentially in the data. Now these names imply that they are actually present, not of course in the sense that the sour-sweet is an *aggregate* of a sour and a sweet that can be separated as much as if they were tasted at different times, but in the sense in which we speak of a *mixture* as opposed to an aggregate. The fact that such a mixture is possible here, i.e. that sour and sweet may be united in one im-

pression in a manner that we can scarcely describe but easily feel, while sweet and red cannot, distinguishes the relation of the several tastes to one another from that of the disparate groups A B C.

It may be objected that in the sour-sweet the difference between the sour and the sweet is only present potentially not actually; that there may easily be a third impression ω, itself simple and in no way compound, yet forming a connecting link between μ and ν; and that this then, on account of its resemblance to both, is designated in speech by the two limits μ and ν between which it falls, without implying that it actually is a mixture of the two. This objection I should not consider sound unless there were present in ω besides that in which it resembles μ and ν an independent remainder that could not be accounted for by the combination of μ and ν; where this is not the case this third impression ω will not merely be called a mixture μ ν by an arbitrary freak of fancy, but will in fact be that and nothing else. But the primary forms μ ν π and all mixtures of these, though made one group by the fact that they all alike appeal to the sense of taste C, yet within those limits can only be regarded as *disparate* from one another. A man who had tasted nothing but sweet could never by any conceivable modification of the feeling it gave him discover the peculiar nature of sour or bitter that he had not yet experienced. There is then no transition from μ to ν or π through independent connecting links, but we must first know μ ν and π and then get the intermediate links by various mixtures of these.

We find the same relations between colours, and I took occasion in an earlier passage[1] to justify the common usage of speech in always distinguishing a limited number of primary colours, and inserting the rest as mixed colours between them. It is of course possible to lead the eye gradually through skilfully selected middle-tints from the

impression of one colour to that of another: but while red passes into orange or violet only by an admixture of yellow or blue which can still be felt as yellow or blue, that which makes red what it is does not pass over into that which makes blue what it is. A man who had experienced one but not the other could never discover in the simple nature of red anything which could possibly be modified, heightened, or cooled down in such a way as to lead him to imagine what blue is: he would have to learn what blue is before he could mix the two extremes together so as to arrive at the intermediate violet. The modifications of which the several primary colours are capable must also be regarded in the same way. We undoubtedly have the right to consider bright blue and dark blue as kinds of the same blue: but these kinds also are produced by the mixture of white or black with a pure blue that is always the same though never visible in its purity. Only I would once more briefly remind the reader that all that I have hitherto said refers only to the nature of our sensations after they have arisen in our consciousness, and has nothing to do with the physical or psychical conditions of the act of sensation.

174. With sounds the case is essentially different. After a comparison of several sounds we distinguish first of all three predicates. The peculiar tone of the instrument which is sounding, whatever the physical antecedents may be, is for our feeling a simple property which defies further analysis, more analogous to a taste than to anything else. However strongly we may be moved by the secondary effects of this peculiar tone the essential nature of the note seems to us to be quite independent of this, and also of its second property, viz. its loudness or strength: we regard both only as ways of producing the same note, the distinctive nature of which lies in its pitch. But in this third aspect sounds do not like colours fall into a number of distinct stages, such that one can pass into another only by mixture, but they rather form a continuous series, in which

the difference between two more distant members is only a multiplication of the difference between two adjacent members. It is impossible to make a proportion in which red shall stand to blue as yellow to any fourth colour: but the difference between two notes can always be stated as a multiple of some difference which we take as the unit. This difference itself is of a quite peculiar kind: we should not use the phrase 'higher' and 'lower' in speaking of sounds, unless, quite apart from the frequency of the sound-waves which we certainly do not feel, our feelings themselves announced one note as a heightening of another: but this quantitative idea cannot be referred here as it can elsewhere to a qualitative content that is independent of it: a note *d* is different from another *c* even in quality just because in it the undefinable common property of sounding which it shares with *c* is 'heightened' in that peculiar way which we can only express by this happy metaphor, or at most by the more technical phrase 'qualitative intensity.' The differences of notes therefore are homogeneous and measurable in extent, which the differences of colours were not: the notes intermediate between two others are not formed by mixing these two together, but are on a footing of perfect equality, as original members of the series, with those members between which we place them.

And lastly the whole series is endless: it is not possible, in addition to the colours known by experience, to imagine a new colour of which we can have an idea though it happens that our eyes never saw it; the scale of sounds, on the contrary, may be continued *ad infinitum* because each is generated out of its predecessor by a heightening which is felt to be homogeneous. It is not unmeaning to talk of sounds higher or lower than any that can ever come within our experience, because we have here (what we could not have if we tried to imagine new colours) a distinct idea of the way in which these sounds would differentiate themselves *if* they were audible.

175. With some modifications, which I leave the reader to make, these remarks apply also to the series of our sensations of heat: but at the same time the latter exhibit a new feature. The living body's own need for warmth gives a peculiar significance to certain sections of the series; we distinguish cold, cool, lukewarm, warm, hot, and fancy that these terms have a definite meaning; but not only would it be impossible to draw a hard and fast line where cool ends for everybody and lukewarm begins, but even if we interrogate our own feelings merely we are obliged to confess that there must be a certain caprice in choosing the one name or the other. We may connect with this contrast of heat and cold, and of high and low sounds, a great number of other pairs of ideas, the content of which is not so directly derived from sensations, e.g. great and small, strong and weak, many and few, old and young, and many more of the same sort.

However decided a contrast is intended by the two terms of these antitheses it is always impossible to mark off the province of the one from the other,—they constantly and insensibly pass into one another. But when we go through such a series the passage from *a* to *z* and that from *z* to *a* are very clearly different,—to some extent they admit of definition, and our immediate feeling at any rate never fails to distinguish them. We cannot say what is warm nor what is cold, but we can say without any doubt whether *a* is warmer or colder than *b*: in this case the decision is a matter of sensation; in passing from *a* to *b* we are conscious of a change which is the opposite of that which we experience in passing from *b* to *a*. We cannot say what great and small mean, but the statement that *a* is greater than *b* is quite free from ambiguity, and may be defined to mean that if *b* is taken from *a* there is left a positive remainder δ. And it is the same with the other examples: these adjectives are all derived not from the apprehension of one idea but from the comparison of several, and denote

relations which have no fixed value or meaning apart from a second point of comparison. These adjectives therefore are indefinite in the positive; only their comparatives have an unambiguous meaning. Where the positive form is used in speech it means that the comparative term may be applied to the thing denoted when compared with an unexpressed standard, which either in the estimation of the speaker or in common opinion is the normal or usual state of the thing in question.

176. There is one more point to consider in connexion with sound and sensations of heat. Sounds being in themselves of perfectly equal value we have no inducement to select some few of them as fixed points and to give them prominence by naming them. But on aesthetic grounds we want to articulate the whole series. As the simple sensation of a note is undefinable we characterise it by stating the cause which will at any moment produce precisely that note, i.e. the frequency of the vibrations upon which it depends. But there is no reason for preferring one number to another, and as every member of the series may be defined with equal ease in the way named, the musical scale has in fact no absolute starting-point. It is true that other circumstances, viz. the harmonic relations of notes, which I must here pass over in spite of the interest which they have even for the logician, lead us to arrange the series in octaves; but even this arrangement has no fixed starting-point; we may begin at any height we please.

Our sensations of heat do not admit of such a simple definition by their causes; we are obliged to have recourse to the other observable effects of their unknown cause, viz. the expansion and contraction of bodies. To take the melting-point of ice as the point from which the degrees of temperature should be measured in an ascending and descending scale was to choose a quite arbitrary zero to reckon from, though one very well adapted to its purpose: for the fluidity or solidity of water is a point of cardinal

importance in the meteoric and organic processes which surround us. But it is after all merely a zero in our calculation, not in the thing calculated. Starting from the unknown amount of heat (call it x) which is present at the melting-point of ice, all we do is to reckon the increase or diminution of this amount by multiples of a unit-degree chosen expressly for this purpose. Thus 12° is not the double of 6°, but the difference between 0° (which is equal to x) and 12° (which is equal to x + 12 units) is twice as great as the difference between 0° or x and 6° (which is equal to x + 6 units).

The reader may see by this simple illustration that though a series or a complex system cannot be articulated and arranged in a regular order unless there be a corresponding regularity in its own relations, yet thought frequently has to take a quite arbitrary starting-point and an arbitrary standard in order to master and make use of this regularity; and that such an arbitrary arrangement, though admitted by the nature of the object and justified in its results, yet must not be looked upon as a property inherent in the object itself.

177. Practical life offers many illustrations of this remark. We here have to do with qualities which either attach to various persons and things in very varying degrees, or which in one and the same subject take successively a continuous series of values, from which proportionate effects are expected. But it is only in nature that effects vary continuously in accordance with the conditions: where the result does not follow till it is produced by human action, the exact observance of the desired proportion is generally prevented by the fact that the labour required would be out of all relation to the end in view. We have to content ourselves with breaking up the whole series of values into sections and acting as if the conditions were the same throughout each section, fixing the result at an average amount, which will be too great for the first and too small

for the last members of that section of the series. Thus for the purposes of taxation we divide the series of properties, from absolute poverty up to the highest pitch of wealth that is likely to be found, into a number of classes; in calculating the premium to be paid on a life-insurance we reckon age by years or at lowest by some considerable fraction of a year; in calculating interest we keep to a day as an indivisible unit. Again it may happen that a quality gradually attains a certain pitch to the attainment of which we desire to attach certain consequences, though we cannot say at what moment the decisive condition is fulfilled. That maturity of body and mind which we have in our minds when we say that a man is of full age or has attained his majority is certainly attained by different persons at different times of life; but it is impossible to find out the actual moment in each individual case, not merely because it would necessarily be an endless business to appraise the total merit of the person, nor yet because such a censorious proceeding would be unjustifiable, but because, though the higher grades of maturity and immaturity are easily recognisable, there is really no certain mark to distinguish them in doubtful cases. But for all that the needs of social life require that a definite time be fixed; so the law has to fix it summarily, and attaches to the completion of certain days and hours the beginning of certain rights and duties, though no one supposes that the capacity and the obligation which were absent yesterday have actually sprung up in the course of the night. But though this proceeding is summary it is not without reason: the choice is limited to times which correspond without any appreciable difference in accuracy to the requirements of the situation; all that is arbitrary is the preference of one out of a number that would all do equally well.

There are other cases in which we are still further from finding any precise standard in the nature of that which has to be settled, and must look for it in the further ends

whose attainment is to be facilitated by the settlement. Such are the fixed periods within which certain conditions must be satisfied in order to establish some legal claim or to avoid some legal obligation: though the outlines of these arrangements are determined by the object specified, their details aim at nothing but logical precision. This our ancestors effected by not measuring the more important periods by entire units of time of the larger kind, but adding to such units some fraction of them, some days to a week, some hours to a day; by these means they narrowed the period within which (to use a common but rather loose phrase) a man might have fancied that he was satisfying the law. The police again are quite right when in order to prevent disturbance of the peace they summarily fix the number of persons that shall be held to constitute a forbidden assembly at three or five, thereby barring disputes like those the old sophists used to raise when they asked how many grains of corn are required to make a heap, or how many hairs must be lost to make a bald-head.

178. To return from this digression:—whether a note is to be called high or low, a liquid hot or cold, are questions that people never quarrel about: there are no interests attaching to the content of these conceptions that could make us hesitate to admit at once that their meaning is, as we said, relative. It is different with *good* and *bad*. We set the highest value on the fixity and absoluteness of these conceptions: every action, not simply as compared with others but as it is in itself, must it is thought be unequivocally included in the one and excluded from the other; people even think they are bound to deny that there are any degrees of goodness in the good or of badness in the bad, for fear lest the diminishing values of the two should at last meet in the indifferent as a zero-point, and a constant transition be thus set up between two opposites which ought rather to be severed by breaking

down every bridge. But this logical rigour is utterly at variance with the unprejudiced judgment which we all bow to in real life. No one really doubts that there are degrees of goodness and badness, and no one can persuade us that no acts are indifferent till he has artificially limited the conception of an act. But it really is no use to try to fend off the threatened confusion of good and bad by first dividing all actions into those which can be judged morally and those which cannot, and then proceeding confidently to divide the former into two absolutely opposed groups, the good and the bad. We thereby do but move our doubts a step further back; for the question now is where is the line to be drawn between that which calls for a moral judgment and that which does not; and this line as before will seem to vanish in a perpetual passing of the one into the other.

Again the relation of the pleasant to the beautiful and the good, though a less pressing question, is one of great interest on aesthetic grounds. To the man without a theory they seem to arrange themselves in an ascending series, not merely according to their value but according to the meaning of their content; not of course in the sense that by mere intensification what is extremely pleasant would become beautiful, or the highest beauty pass into the lowest grade of goodness, but in the sense that there are kinds of the pleasant, distinct in quality, which begin to have a right to the name of beautiful, and forms of beauty which produce an aesthetic impression akin to moral approbation. But those who theorise upon morality and upon art alike resist this admission; they deem the beautiful falsified if it has anything to do with the good, the good degraded if it has anything in common with the beautiful, and through this with the pleasant. Here too, with regard to beauty at least, people have been found to deny all differences of degree, and to maintain that what is beautiful at all is entirely beautiful, and that if you allow there is

anything more beautiful, you cannot think this really beautiful at all.

179. Let us, in order to settle these doubts, look around for other illustrations. Of the straight line, from its nature, there is of course but one species known to the geometer; but in curves he distinguishes countless degrees of curvature of measurable value, so much so that the straight line itself appears as the extreme limit to which the curve constantly approximates as the radius increases. Yet in spite of this unbroken continuity not merely does the geometer persist in the general statement that curved and straight are opposites that can never be reconciled, but no doubt ever arises in its application to a particular line which is accurately known; however near it may come to a straight line it is yet quite undeniably curved, so long as the radius of curvature has any finite magnitude.

Again a curve may in one portion of its course be concave to an axis to which it is convex in a further portion; if it makes this change of direction in an uninterrupted sweep without any angle that breaks the continuity, there is no doubt that its tangent at the turning-point, and therefore the element of the line itself, is parallel to the axis in question, and so neither concave nor convex; but although both directions thus visibly meet in one zero point of indifference that belongs to neither, yet the opposition between them is thereby neither altered nor removed; on this side of that point the curve remains entirely concave, on that side entirely convex. Take a simple instance: between 1 and 2 we may insert countless fractions rising gradually in value from 1 to 2; between full daylight and midnight darkness countless degrees of illumination not only are conceivable but actually occur; between pleasure and pain there lies an uninterrupted series of feelings which connect the one with the other: but 1 does not on that account become equal to 2, nor do darkness and pain cease to form a perfect contrast to light and pleasure; and

at the same time each member of these pairs, by itself and without reference to the other member, is something so definite that we never mistake the one for the other. These illustrations are sufficient to explain the statement that the existence of countless degrees through which two opposites A and B pass till they meet in a common zero-point of indifference, does not destroy the difference or opposition between the meanings of A and B themselves.

180. And so even if the moral philosophers had succeeded (and it is their business and not ours at present) in determining what they *mean* by good A and bad B as precisely as the geometer defines what he means by convex and concave, they would still have had no ground for denying that good and bad have degrees and meet in the indifferent, in order to maintain unimpaired the distinction between the two. The specific meanings of the general conceptions good and bad are not in the least degree altered because particular cases to which the terms are applied partake more or less fully of the character of one or the other of these opposites. But the zero-point of indifference can still less contribute to the confusion of the two, for its meaning is not that both are true at this point, but that neither is true; it is therefore merely a point of separation,—on this side is only good, on that side only bad.

On the one hand then the maintenance of the distinction between good and bad is no reason why people should deny that there are degrees of good and bad; on the other hand we must insist upon an explicit admission of the fact that there are degrees. To deny it, to repeat the old Stoic paradox *omnia peccata esse aequalia*, or to go on preaching that even the smallest error is still not truth but error and nothing else, is but to waste time in tedious assertions which as they contain only half-truths may on this very principle be called errors and nothing else. It is not true that a curve is once for all a curve, so that the degree of its convexity or concavity is quite a secondary consideration,

which has nothing to do with its character as a curve; the fact is that one curved line is actually more curved than another, and so realises more intensely the character common to both. Similarly the good or bad intention out of which an action springs can not only be measured in a secondary way by the importance of the interests affected by the act or of the circumstances under which it is done, but can itself be estimated according to its degree of goodness or badness; for such an intention is by no means a mere form which is alike in all cases; it is an inner process which not only must reach a certain degree of intensity in order to generate the impulse which every act requires or to overcome certain obstacles, but has also a certain degree of value according to the amount of the good or evil which it consciously aims at producing. Error again is not merely not-truth; that would not distinguish it from doubt; it is a departure from truth, and has therefore a measurable magnitude, indeed is inconceivable without it; a man whose thoughts are occupied with real problems therefore will not be so silly as to reject in identical terms, as mere errors, two assumptions of which the one is so far from the truth that it leads to no knowledge at all, and the other so near that it leads to nearly all the knowledge of the subject that can be expected.

181. It may be that the series of the pleasant, the beautiful, and the good (the further consideration of which I leave to the reader) has already suggested another relation that can exist between a series of conceptions, which I will first of all illustrate from geometry. Imagine two pyramids *A* and *B* presenting similar horizontal sections but one sloping more steeply than the other; if we place them so that the apex of the one (the less steep) lies within the other (the steeper) and upon a point in its axis, then the plane which passes through the intersection of their surfaces belongs both to the series of planes of which *A* is the integral and to the series of other planes the endless succes-

sion of which is summed up in *B*: similarly we can imagine a third pyramid *C* which should in like manner have a plane in common with *B*.

Now the generating law of each of these solids, with reference to the common axis of all three and the position of the apex in that axis, may be stated in a formula, which would have to be compared with the general conception of *A B* and *C* respectively. It would then appear that in the *A* series there is one member that also satisfies the requirements of *B*; and therefore that as to this member it is a matter of doubt or of indifference whether it is to be classed under the conception *A* or *B*,—not because it satisfies neither, but because it perfectly satisfies both at once. But with the exception of this particular case all the instances of *A*, all the other planes by which the compound solid thus formed could be intersected, would belong exclusively either to *A* or to *B*. The same would be true of the plane common to *B* and *C*.

In these cases then it is due to the very nature of the essentially distinct conceptions that certain members of the series which they severally characterise become ambiguous, so that by themselves and without taking count of some secondary point, such as the manner of their origin and development, it is not safe to ascribe them exclusively to any one of these conceptions, though apart from these particular cases there is no doubt at all about the difference of the conceptions. We have here named *A B C* and so expressed them as conceptions, leaving the particular cases unnamed. But the purposes of speech may sometimes suggest the opposite procedure. We may name and fix certain conceptions *M N O* which have quite unambiguous and distinct meanings only in particular cases, which we may picture to ourselves as salient points, as *maxima* or *minima*, in a connected series. We shall then find the reverse of what we found just now, i. e. we shall find many contents furnished by feeling and experience which have a

place indeed between two of these conceptions, but only between them, corresponding completely to neither.

182. As illustrations of the latter procedure we may take compound conceptions got by starting not from one but from many points of comparison at once. With such a conception no doubt every instance agrees which in each of these respects is found to have the appropriate mark; but the applicability of the conception becomes doubtful in many other cases, which from one point of view would certainly be included under it, but from another which must also be considered would certainly not. Various thoughts thus cross one another in the conception of illness. Illness is certainly above all things a departure of the bodily condition from a supposed fixed standard. But a malformation, which departs considerably from the natural structure of the body, still cannot be called an illness, so long as it does not impair the vital functions, nor so long as it remains constant and runs no natural course through various stages. A wound always in some degree alters structure and function, and also runs a natural course; but a slight wound is not called an illness, plainly because it does not involve danger nor make the body unserviceable for any important purposes of life; but again a very severe wound is also not called an illness though it does both; its origin is too sudden and too entirely due to external violence,—and now we observe that when we spoke of illness, we thought of a state which, though dependent upon some external cause for its origin, yet takes its definite shape from the peculiar interaction of the internal forces. But now a cold is such a reaction of the internal forces against an external stimulus: but a cold is scarcely called an illness so long as the element of danger is absent: and just as we here help ourselves out with the milder phrase 'unwell,' so we use the term health with a certain latitude, allowing room for the slow advance of a number of disturbances connected with individual idiosyncrasies.

It is not difficult to say what is the right course here. It is impossible in such cases to find a definition which shall be in harmony at once with the requirements of science and with these strange caprices of language: if we want to determine the conception, we must disregard usage and fix it arbitrarily. In the instance we have chosen this is scarcely needed, for pathology gets on very well without any unimpeachable definition of the nature of illness in general; and the physician has absolutely no need for logical generalities which yield no guidance in practice.

But in other cases it is not so. In our conception of crime all sorts of consideration cross one another,—we consider whether it was deliberate or precipitate, what was the degree of evil intention, whether it was attempted only or perpetrated, what was the amount of harm done: the distinction between the creations of art and the products of manufacture, or the relation of a free reproduction to a literal copy, presents similar ambiguities. To fix the limits of the conceptions is of more importance here, since by the operation of law certain advantages and disadvantages follow regularly and directly according as a given case is judged to belong to the one or the other; but here also, though we take count of common usage, it is yet necessary in the main to distinguish them by positive enactment.

183. Obviously we may set down any conception M as equivalent to any other conception N when we have by further specification so changed N that it is equal to M. Thus there arise a number of incidental aspects or variations of the expression for the same M, which we shall further on find to be of use in enabling M to be subsumed now under this law and now under that, such law leading to a new assertion about M. There is no limit to the extent to which this procedure may be legitimately carried so long as the transformed M really coincides with the original M, so long that is as N is equal to M. We may even bring a triangle M under the conception of a

four-sided figure *N*, provided of course that we add that one of the four sides is reduced to nothing. This may seem mere trifling, but it is useful in practice: we can thus for instance easily picture to ourselves how every time that two sides of a polygon, which were before separated by an intervening side, are made to meet at their extremities by the vanishing of the intervening side, the sum of the angles of the polygon (in this case four-sided) is diminished by two right angles.

This use of transformation will engage our attention further on; what I wish here to emphasise is that the difference between the two conceptions thus brought together is of course not altered by it. The four-sided figure remains just as distinct from the triangle as it ever was, i. e. so distinct that it must be stripped of its very essence before it can be ranked with the other; and similarly the alterations, whatever they be, that must be made in order to turn *N* into *M*, give the measure of the *abiding* difference between the two conceptions. When we are dealing, not as in this case with abstract constructions of thought but with realities, which have an independent origin in the region of fact, such transformations have very little value; they are in the first instance mere fancies, whose significance cannot be ascertained without special enquiry. In thought we may change any given form of crystal into any other that we please by cutting off slices here and there,—by successive alterations of outline we may change the likeness of a crocodile into that of a bird,—from any one chemical element we may in thought derive all the others by giving successively certain other values to the coefficients which the fundamental properties of matter take in the case of that one. But by such devices we cannot make the conceptions *M* and *N* approximate to one another, for their difference remains always as great as the number of steps that we must take to get from one to the other; neither can we

thus establish between the actual things which exemplify these conceptions such a connexion that one might pass over into the other. For that it would be necessary to prove that the physical forces of the elements which build up an actual crystal of the form *M* are such as to make it possible for the same elements to be also in equilibrium when arranged in the form *N*; or that the concatenated system of forces which determines the structural type of the crocodile and maintains it in life may be so modified by other natural influences that the form of a bird may actually grow out of it,—that in short the order of nature actually contains impulses which realise the changes which we may choose arbitrarily to make in thought or upon paper. We cannot but remember, though happily as an error which we have outgrown, the wild caprice with which not long ago people would derive a word in one language from any casual word in another, and call it etymology; at the present day people need to be warned against proceeding in a similar way to satisfy the newly-awakened desire to conceive all the various kinds of organic beings as evolved from one other, all fixed specific differences being done away. But, whether Darwin has succeeded or not in his attempt, we must at any rate allow that he has taken the greatest pains to point out the real processes of nature by which the transformation of one organic form into another which we can conceive in thought may have been actually brought about.

CHAPTER III.

Schemes and Symbols.

184. In this chapter I shall continue to treat of the same subject as in the foregoing, but from a somewhat altered point of view. The extent and importance of the difference between several ideal contents can, we ascertained, be precisely determined only when we find ourselves able to compare several differences of the same kind, i.e. when the ideas to be compared themselves form series, whose members proceed according to a law that can be more or less exactly stated, and when moreover from the nature of the feeling whose modifications, distinct both in quantity and quality, are represented by the members of the series, such modification can only take place in one and the same direction. Compound conceptions whether of things or properties, situations or events, by reason of the number of the characteristics or of the aspects which they include, may be altered in various directions; one or some or all of these characteristics and of these aspects may run through all the various phases of which they are capable; and again the bonds which connect them may pass through all the various degrees of laxity and strictness and all the changes of form to which they are by their nature liable.

Now there is no reason why the value or the extent of the difference between two such compound conceptions M and N should not frequently be revealed to us by a direct impression with as much certainty as we need require in

the case in question: if however a more accurate determination were needed for scientific purposes, we should have first to determine the values of the various scales upon which the several alterations take place, and thence to determine the value of the total alteration which separates *M* from *N* or *N* from *O*. The reader may be inclined to object at once that in most cases at any rate we proceed in the reverse order to estimate the significance of the scale of a change which has taken place by the amount of the change which this alteration has produced in the total impression. I may allow this objection without taking any further notice of it; for what I here wish to illustrate is not a logical rule but a propensity of our reason, which needs to be checked rather than to be indulged, but which as it is ineradicable needs to be specially mentioned. It is easy to understand, I mean, how out of the above-mentioned problem may arise the wish to have a universal *scheme* in which not only all the modifiable relations of different elements that we can think of, but also the values of the difference between any two modifications should be laid down so completely that the difference or the kinship between any two conceptions *M* and *N* should be exactly indicated by their position in the universal scheme.

185. To illustrate this I will first go back to remote antiquity, to *Pythagoras*. To reconstruct a body of genuine Pythagorean philosophy out of the scanty and for the most part very questionable materials at our command is a task which I will not undertake, but I think I am able to state what *may* have been the fundamental idea which animated it, and which would enable us to understand why the sympathy stirred by it has been so lasting though often so perversely expressed. It is tolerably certain that the bent of the school was first to abstract mathematics, and secondly to their application to the processes of nature. The first line of study could not fail to lead them to picture the series of numbers and the world of shapes as two great

coherent systems, and further to bring them to see how spatial figures themselves depend upon the numerical magnitudes which they involve. The second, besides other less known results, led to the discovery of the relation between the pitch of a note and the length of the vibrating string, and thereby no doubt suggested the general idea that even phenomena whose differences are in the first instance felt by us as differences of quality are based upon mathematical differences that admit of comparison. The rash generalisation of results thus won is what the fancy of men is always prone to; the mathematically-trained Pythagorean went so far as to make the reflexion that if it be once established that a series of changes in phenomena corresponds to a series of changes in magnitude, then every other conceivable mathematical relation along with all its modifications must have its counterpart in the phenomena, —or conversely, if a group of phenomena is based upon definite relations of magnitude, the coherence of all the processes of nature necessitates the conclusion that all other phenomena also depend in like manner upon relations that can be mathematically determined.

This I conceive to have been the origin of those speculations which Aristotle expresses by saying that Pythagoras regarded the principles of numbers as the principles of things: but we must further consider the meaning of this expression. The purport of the Pythagorean philosophy was certainly wider than we might be led to suppose by that other saying of its author, that God has ordered everything by measure and number; i.e. it was not limited to the mere application of mathematics to nature, if that means merely that the definite magnitudes of natural forces and processes modify one another when brought into contact according to the same mathematical laws that hold good for magnitudes in general: these data themselves, to which mathematics are merely *applied* by modern 'mathematical physics,' were regarded by Pythagoras as in themselves form-

ing a system whose inner articulation is based upon the same relations that determine the structure of the series of numbers and of all their possible combinations. I wish to distinguish in this theory a general idea and the particular form given to it.

186. The so-called natural philosophy of the Ionians had devoted itself to describing the processes by which natural bodies were formed out of their primitive matter and returned to it again. As this philosophy very generally used for this purpose the ideas of condensation and rarefaction, it may appear, in virtue of its employment of quantitatively determined means, to be closely akin to the Pythagorean theory. The two are nevertheless very far apart: for the Ionians never betray any desire to show that the sum of that which is thus produced at any moment of its existence or in the whole series of steps by which it comes into existence forms a coherent whole of mutually dependent parts. Pythagoras on the other hand seems to have troubled himself very little about this *origin* of the world, but the world as it was after it had come into existence was to him a system, such that not merely were its parts there, one beside the other, but that there would have been a gap in it if while one phenomenon were present another had been absent. If *a* and *b* and *d* are present, then if *c* is there at all, it is not merely there along with the others, but it is there because the law according to which the series *a b* advances to *d* requires it as the third member of the series which is indispensable to the presence of the fourth member *d*: or if *c* is absent, it is not merely absent as a matter of fact, but because the law which regulates the series excludes the possibility of this third member before *d*. The same consideration may be applied to other series in the actual world, to $\alpha\ \beta\ \gamma\ \delta$ and to $\mathfrak{a}\ \mathfrak{b}\ \mathfrak{c}\ \mathfrak{d}$, and this application was made by the Pythagorean school.

How they conceived the relation between the different characters of these series, which I wished to indicate by the

use of different alphabets, is a point upon which we are certainly in the dark, and upon which, as we may gather from Aristotle, the fullest information would probably throw but little light; but with respect to the law which in each of these series binds the homogeneous members together, it seems to be indubitable that it was regarded as precisely identical for all the series, i.e. that they maintained a complete parallelism between the relations prevailing in the various groups of connected phenomena. This is shown in the supposition that the earth has an invisible fellow, in order to bring the total of the then known planets up to ten, to which number the arithmetical mysticism of the system had once for all assigned a peculiar significance,—in the assumption of a fifth element, which together with water, earth, fire, and air, shall correspond to the five regular solids, tetrahedron, cube, octahedron, dodecahedron, eicosahedron, —in the attempt again to conceive the distances of the planets as arranged according to musical intervals,—and even in the meagre form of their tables of opposites. To us of course these tables do but illustrate the frequent occurrence of this relation of opposition between two conceptions even when these are arbitrarily chosen, but the fact that they always contain ten pairs seems to indicate that they were intended to represent this relation as essential for all the different stages in a series of ten members. Finally when they assigned life to the number six, intelligence and light to seven, and friendship to eight, we see that they regarded not merely the phenomena of nature, but also those of mind, and in a word every conceivable thing, as ordered according to the same serial law.

This philosophy then sought and fancied that it found precisely what we spoke of above, viz. a universal scheme which mounting from simple to complex was supposed to embrace the whole sum of possible forms, one of which was to serve as a pattern for the formation of every actual thing, while at the same time these forms or types were to be so

arranged in the scheme that the position of its type directly determined the significance of every actual thing, and the amount of the difference or the kinship between it and other things formed upon the model of other members of the series. The general idea then that I would ascribe to the Pythagorean philosophy is this, viz. not merely a subsequent arrangement of things whose nature was originally settled without reference to the principle of this arrangement, but a harmony of the Cosmos—which name was first applied to the world by Pythagoras—based upon the notion that all things are from the beginning nothing but various realisations of a series of types, regulated by one law of development which is the same for all.

187. The general conception is undeniably grand, but grandeur is sadly lacking in the special form here given to it. Even in the present state of the mathematical sciences, various as are the magnitudes whose interesting mutual relations have been examined, it would be impossible to find adequate types or symbols or abstract expressions for the still more various relations that subsist between the elements of the actual world and the combinations that arise out of them; but the arithmetic of the ancients, which the Pythagorean school seems to have helped to develop, furnished in its then state but very few and very meagre numerical relations, whose significance must have been much exaggerated and from the beginning very arbitrarily interpreted before they could be regarded as the relations upon which the structure of the world is based. The grounds on which they justified their well-known veneration for the number ten,—viz. the fact that all numbers are generated by the repetition of unity; that in this series the even numbers alternate with the odd numbers, which cannot be divided by 'the principle of multiplicity,' i.e. by two, and which are therefore held to be of higher rank; that three is the first union of odd and even, four the first square of a multiple number, and ten the sum of these

exalted four first numbers,—are grounds which could not be admitted except by a system of symbolism which was ready to accept any *interesting* motive without regard to its connexion with others: though the real grounds of that veneration undoubtedly lay in the habitual use of the decimal system. If these thinkers had been acquainted with all the algebraical and transcendent forms of functions which are the instruments of modern mathematicians, how much more various would have been the symbols employed, and how much more delicately would they have been adapted to the nature of the several phenomena! The same tendency still survives in us: even in cases where calculation in the strict sense is impossible we are inclined to use the term 'power'[1] when the meaning and importance of a conception is raised in some peculiar manner, as for instance when each of the centres of relation, whose determination by each other constitutes the meaning of the conception, is itself exalted into a small system, whose members determine each other in the same way.

We can imagine then how the Pythagoreans (if they had had our knowledge) might have illustrated many relations of dependence between various elements by the relation of a logarithm to its number, and how they might have applied trigonometrical functions to explain any kind of periodicity. As however they had not our resources at command, and as even these would still be insufficient, it would be quite useless to examine in detail the reasonableness of the Pythagorean symbols.

188. That it was the fate of the whole theory to be variously interpreted and misunderstood is easily explained by its nature. According to one statement of Aristotle it was the *principles* of numbers that Pythagoras identified with the *principles* of things. This seems quite intelligible. By these principles of numbers must be meant the relations between one and the other numbers, the way in which one

[1] [In the mathematical sense.]

can be repeated, the divisibility or indivisibility of the rest, —in a word the possibility of generating the whole series of numbers by the use of these constant relations and operations, or, as we should say, the possibility of exhibiting every number as a function of other numbers. Things then, ought also to have the same inner structure, their series ought also to be arranged according to the same principles, so that the nature of the one might be exhibited as a function of the nature of the other.

But it is also asserted by Aristotle along with others that the Pythagorean school declared that numbers were things, or at any rate that things were numbers. Even this is quite intelligible to any one who is acquainted with the history of philosophic ideas and the customary ways of expressing them. To a certain extent indeed the Pythagoreans would have been right in making this assertion, and this justifies us in supposing that they actually made it; for as already said what they intended was by no means merely to apply numbers to the quantitative determinations of things whose real nature is independent of these determinations,—e.g. you may have similar triangles of very various sizes: their numbers were meant to signify that which distinguishes the essential character of one thing from the essential character of another; *a* was *a* because its content was constructed according to α the function-form or the generating law of one symbolic number, and was thereby distinguished from *b* which was *b* because it followed β the generating law of another symbolic number. It was quite possible then to say, with a reservation to be presently noticed, that the essence of a thing, in the sense of that which distinguishes it from another thing, lies in the number immanent in it.

The other assertion that the essence of things, in the sense of that in virtue of which they all are things, or their reality, consists in these numbers, or that numbers are the real things, was perhaps not positively made by the Pythagoreans in this form: if they did make it, they certainly

could not justify the latter expression, but they could assuredly justify the former: for if there is actually nothing whose nature is not determined by one of these symbolic numbers, the numbers are assuredly the *conditio sine qua non* of every reality; to treat them as more than this, and to speak of the numbers themselves as the real things, is an unwarrantable straining of language, though we shall presently see how prone to it the thinkers of all ages have been.

There remains one great imperfection which we have already mentioned. The same typical series of numbers has to repeat itself in a number of parallel series of actual things, in $a\ b\ c\ d$, $\alpha\ \beta\ \gamma\ \delta$, $\mathfrak{a}\ \mathfrak{b}\ \mathfrak{c}\ \mathfrak{d}$; how then are the members b β $\mathfrak{b}$ distinguished from one another if the whole nature of each of them is exhausted by the same symbolic number? To this there is no answer possible: at this point the theory, which aimed at embracing the nature of things completely, relapses again into a mere application of a general law of structure to various cases whose characteristic differences must be regarded as given. But this is what makes it serviceable for our present purpose as an illustration; it thereby becomes an attempt to frame a universal scheme for the relations of kinship and difference between all the groups formed by kinds of content that can ever by any possibility come to be considered.

189. In order to justify the length of this discussion I would point to the extraordinary tenacity with which this desire to find a scheme for the whole contents of thought has maintained itself through the course of ages. It showed itself first in this form of mystical speculations about numbers; over these we may pass very lightly; as such speculators were satisfied with anything however meaningless so long as it was interesting and startling, they were, to speak plainly, always in search of a secret truth which they never found, and it must always have needed a very sym-

pathetic hearer to find in the symbols a better expression for the meaning put into them than could have been obtained without them.

Presently the speculators ceased to found their dreams on this purely arithmetical basis and wandered away in various directions. In the first place every discovery made by advancing science that has any important bearing upon the relations of things has almost without exception been extended into a scheme for the articulation of the whole world. For a long time people traced everywhere the behaviour of the four elements of the ancients; and in later days the mystic significance of this number four did not pass away, it was only transferred to the newly discovered constituents of organised bodies, carbon, hydrogen, oxygen, and nitrogen; it agreed admirably with the four quarters of heaven, for zenith and nadir of course fall outside our natural line of sight; it agreed equally well with the four seasons of the temperate zones, within which these speculations were carried on, and with the four indispensable cases of nouns; at a later date, as the theory of astronomy came to completion, the contrast between centrifugal and centripetal tendencies entered into men's notions of all things and was fused into one with the opposition of the sexes and the relation of acid to alkali; the discovery of magnetism and electricity caused the scheme of polarity to be carried even further if possible into the consideration of all conceivable things.

Other speculators proceeded in the opposite direction, starting from the just reflexion that even the relations of numbers are, in part at least, only instances of other still more abstract fundamental relations; these then (they hold) must be sought, and will be found if we simply reflect upon the operations by which our intellect does in fact arrive at its ideas of all things whatever. Now every idea, or at least every compound idea, is made by setting down an *a*, distinguishing from it or opposing to it a *b*, and finally bringing

both into a relation c; thus thesis, antithesis, and synthesis come to be regarded as the scheme upon which all reality is constructed and as the rhythm which thought must maintain in the orderly consideration of that reality. But it is easy to see that the more abstractly these symbols are conceived the more they pass over into *notiones communes* which do indeed apply pretty well to everything but give us no adequate knowledge about anything. Logic then meets all this wild talk with the demand that things be considered, divided, and investigated simply and solely with reference to their several natures, for there is no universal scheme that can be applied, and the employment of merely fanciful models can only injure the impartial quest of truth.

190. Of this unfavourable verdict I can abate nothing, and in some remarks which I wish still to add I have no such intention. When the content M of a conception, an idea, or a perception is given to us in such a manner as to unite in the form μ a number of characteristics, or parts, or points of relation, it is a quite justifiable scientific curiosity that prompts us to enquire how the examples of M will behave, how they will be altered and distinguished from one another, when we vary within the allowable limits either the parts of M only, or both them and the general form of union μ.

In the first place if we keep to the former kind of alteration, there will usually be but little interest in tracing all the kinds of M that are got by simply changing the quantity of the characteristics, for these kinds will, in most cases at least, resemble one other and only repeat the same thing on a different scale. But if one of these characteristics m be of such a nature that for it the opposition of negative and positive has a plain and palpable meaning (such an opposition for instance as there is between right and left, attraction and repulsion, concave and convex, and generally between ascent above a zero-point and descent below it)

then it concerns us greatly to know what happens to M when we substitute $-m$ for $+m$ in its generating law. Supposing $y = fx$ is the equation of a curve, we always take the trouble to set down in turn the positive and negative values of x, and not till we have united the results thus obtained do we think we have arrived at the nature of the curve, which in this case presents itself to our perception not as a mere generality, but as the *whole* which is got by combining every possible example of the general equation. If we happen to see, in a piece of ornamentation, a volute which bends downwards to the right, our imagination is stimulated in a similar way; even if we have no mathematical knowledge of the generating law of this curve, we understand, by reason of the homogeneousness of directions in space, that the volute might be repeated in a precisely similar though opposite bend upwards to the right, and again with another opposition upwards to the left and downwards to the left. If now these continuations, suggested by the beginning which we see, are not carried out, though the surroundings do not give any obvious reason for this incompleteness, our aesthetic feelings are unsatisfied, but this demand for symmetry has also a logical foundation. It is of the very essence of a law that it shall apply to all variations of the points of relation which it comprehends; there is therefore a contradiction in a perception which suggests a law together with the possibility of its prevailing universally, and yet actually presents it as prevailing only in part: what we miss in the perception appears as a defect in the thing: we supply it in order to remove the groundless want of universality.

We always feel a similar impulse in examining conceptions. Whenever in any M one of its determinants may vary from $+m$ to $-m$, which it can only do by passing through the intermediate value $m = 0$, the tripartite division thus suggested becomes for us a *scheme*, which we take as the basis of our investigation of the whole extent

of M. This is the point which I wish here to emphasize, in order to mark the difference between this proceeding and the wild dreams we have just condemned,—viz. that this scheme can be nothing but *an invitation to turn our enquiry in a particular direction*, and cannot give us by anticipation a picture of the result at which we shall arrive. It does not always happen, as in the case of the volute, that the counterparts we expect can be found: whether the change from $+ m$ to $- m$ gives other possible kinds of M at all depends upon the nature of the form of union μ. Still less can we see beforehand whether the kinds thus obtained will be in any way proportional to the differences of the conditions, and if so in what way: it is quite possible that for a certain μ this absolute opposition of $+ m$ and $- m$ is absolutely meaningless. Our method then will be to let μ likewise pass through all the possible forms given by the various alternatives; here also for mere additions of quantity we shall expect only a series of similar results, but for every cardinal point at which μ takes a qualitatively different significance or passes at a bound into its opposite we shall expect a quite new formation to appear in M which depends upon μ; and lastly for every remarkable feature which we find in a special case of M we shall expect to find as counterpart an equally remarkable feature in a similarly conditioned special case of a similarly constructed N (as for instance when we find that waves of light behave in a certain way we look for corresponding behaviour in the waves of sound): but all this remains only a *question* put to the object, to which we await the answer: the answer which enquiry yields may turn out quite contrary to what we expect, but must be accepted whatever it be. Where those dreamers deceived themselves was in supposing that whenever their scheme which they assumed to be universal was applied to any matter whatsoever, every place in it would always be filled by some remarkable form of that matter, none would ever remain empty, and further in

supposing that as these various matters, passing through the same sequence of changes, filled up the several places of the scheme, the forms which filled the same places would by a striking resemblance or analogy in their whole character announce themselves as connected, as akin to or as counterparts of one another. When this was not the case, there was a strong temptation to try to fill up the gaps by groundless suppositions, and to restore the desired symmetry in the corresponding members by giving undue prominence to secondary features.

191. Among modern attempts to unfold in a scheme the meaning of the world there have been some grand ones which even seemed to avoid an essential fault of the Pythagorean theory. In another work ('Geschichte der Aesthetik in Deutschland,' p. 176 ff.) I have examined at length the motives which led to the development of the Hegelian dialectic, the most important of these attempts; I will content myself here with making a few remarks on its logical character. The Pythagoreans in conceiving development in countless parallel series with different contents took no count of the differences by which the corresponding members of the various series are separated from one another in spite of their occupying the same place in the general scheme. The decimal system, with its ascending powers of the number ten, never led them, as it might well have done, to treat these parallel series as themselves successive periods of one and the same main series, resembling one another in their internal structure, but raised one above the other so to speak by the height of the level at which they exhibit this structure, like the octaves in the musical scale.

The imagination of the modern philosopher has supplied this deficiency; the many parallel series are contracted into a single series, composed of cycles of similar structure, the last member of each cycle making a starting-point of a distinctively new character for the development of the next.

If it is possible to find the first member of the whole series and the law which determines the form of the first cycle, the variety of the contents which form the members of the following periods may be explained by their distance from the starting-point and the transformation which the initial member has undergone at each step of the way. Hegel then requires us to concede as a metaphysical presupposition, of whose correctness logic cannot judge, that the world is no sum of things that stand and events that go on one beside the other, the former standing quiet till they are stirred to change by a stimulus from without, the latter determined in their inter-action and in their whole course by universal laws that hold good always,—but that instead of this all the variety of the world is only the development of a unity that never rests, all events only stages in this development or secondary effects of it, and things themselves but appearances, either transitory or begotten anew at every moment, whose whole being lies in the active movements of that unity, crossing each other and coming to a focus in them as subordinate vehicles of that development.

In this account of Hegel's point of view I make no pretence to unimpeachable accuracy, which it would be difficult to attain in a long exposition and quite impossible in a short statement; but what has been said is enough to enable us to understand that within each dialectic cycle these different forms, whose significance somehow constantly increases, cannot simply occur one beside the other, but that each must issue out of the preceding one: development, in short, is the very essence of the system.

192. Now no development is imaginable without a definite direction which it takes in contrast to others which it does not take; but it is equally clear that in this case above all others it is impossible for the unity which develops itself to receive this direction from without; it must be determined by the nature of that unity itself. But here we find that no accurate and exhaustive expression can be obtained for the

entire nature of that which under the name of the absolute is regarded as the one basis of the world, but that what we mean by it in a sort of presentiment is fully revealed to us, nay comes to be completely itself only in and through the development,—indeed, the very name indicates this, for as it is nothing but development, it cannot be itself before it has begun to develop.

The only point of departure then that is left for us is this fact itself, i.e. the knowledge that the absolute is not rest but development. Assuredly then its development must take that direction and form which follow from the conception of development itself, and which therefore must recur in every example of the conception. This opens up a very simple line of thought. If any *A* is to develop itself, it cannot already be that into which it has yet to expand itself; neither can it not be, or be void of content, for then it would not be the determining ground of that which is to be; as yet unexpanded and shapeless it must still be the determinate possibility of its future growth,—in a word it must be '*in itself*[1]' or potentially that which it is to become. But its nature would not consist in development if it were to abide in this potential state; it must actually become that which it is its nature to be able to become. But becoming or the process of development is only an intermediate step between possibility and fulfilment; as merely coming to be, hovering between starting-point and goal, that which is developing itself would be neither identical with itself as it was in its potentiality, nor yet already that which it has to become. This at once enables us to see why the second stage of the development, in which that from which we started is as it were divided against itself, was called by Hegel 'other being' or 'being otherwise[2]'; we see it still more clearly when we remember that it is to the ground of the whole universe that this unfolding is in strictness ascribed; the process of its becoming does not

[1] ['An sich.'] [2] ['Anderssein.']

consist in a simple movement in a straight line, but in the generation of an infinite variety of forms, of which it was the possibility; each of these is one of its results, none expresses its whole nature; the sum of all may indeed contain a complete expression of this whole nature, but only for the observer who adds up the sum and combines this manifold into a unity in his thought. But that which is developing itself must be this unity not only for others but for itself, if it is actually to become that which it was its nature to become; and thus the name of 'being for self[1]' is given to this third stage of the cycle, signifying the completion of becoming, the attainment of the end of development, the return of the potentiality into itself. This return of course is not a simple return; i.e. we do not mean that the intermediate stage of the process is set aside[2] without leaving any result behind or wiped clean out; it must be set aside in the sense of being stored up and preserved; the last stage, being for self, is richer than the first, the potentiality, by the history of the process through which it has come into being.

It is easy to find images for this; thus the octave of the initial note is a return of the latter into itself, and yet preserves in its heightened pitch the result of the intervals through which it has passed; thus when a mind, in which universal truths were innate in the form of methods which its thought instinctively followed, had, by passing through various experiences and enquiries, involving doubt and the removal of doubt, arrived at a full consciousness of these truths, it would merely have returned to itself and yet would be enriched. I will forbear however to explain in detail the peculiar meaning of these phrases; for us it is enough that in the third stage of the development something is given which is indeed a consequence of the first stage, yet is not identical with it but opposed to it as actuality to possibility.

[1] ['Fürsichsein.'] [2] ['Aufgehoben.']

Thus understood the three moments or stages of 'being in itself,' 'other-being,' and 'being for itself,' are but the component parts of the conception of development, and we shall be able to recognise them in everything that develops itself. But Hegel's system rests, as we said, on the conviction that the whole content of the universe, the whole intelligible world, i. e. both nature and mind, are but stages in the development of the one absolute, and that within each of these great provinces the several members proceed in the same rhythmic order, each founded upon and issuing out of that which goes before, and that accordingly the sum of all that is intelligible and all that is real would present itself to us if we knew it completely as a great series, whose several periods are similarly constructed but have each a peculiar significance in its content which is ever rising higher and higher. Upon this conviction we do not here intend to pronounce any judgment; but it remains for us to ask what is the logical value of the dialectic method just described.

193. It is easy to see that it is not strictly speaking a method in the sense of a direction how to find something that we are in search of; it is rather a *scheme*, in the sense in which we have used the word above, which only invites us to enquire if anything is to be found in a given direction or in a spot already marked out, and if so what it is, though of course it implies a confident expectation that the search can never be in vain. If we try to apply this scheme to the independent treatment of a generic conception *M*, in order to arrange its various species in a series corresponding to their essential resemblances and differences, or if we try by means of it to exhibit in their true relations to one another a series of conceptions which are connected by a variety of other circumstances (as e. g. right, wrong, crime, and punishment are connected), we at once find how uncertain it leaves us as to the direction in which our thoughts are to be turned. It is possible that this uncertainty might

vanish if we could appeal to a complete philosophy which had already set down in a universal series the history of the development of all that is thinkable, and had therefore arrived at a conception of right so perfect as to reveal at once the direction of its further dialectical development. But to say this would be to deny from the beginning the applicability of the method as a universal direction for the discovery of truth; it can prove itself such only by this independent service which we require; i. e. it must be able merely by means of its form of procedure to teach us how to develop any given conception in all its proper consequences.

Suppose then that we have given us the general conception of right, for evidently the other three that we named refer to this as a primary conception already fixed: what now is it 'in itself' or potentially? into what 'other-being' does it pass over? into what 'being for self' does it return? It is at any rate evident that a right involves an estimate of relations which prevail between the claims of various persons to exercise their wills upon some object which brings them into collision. It follows that there can be no right if there be no world with relations and objects for the exercise of will, or if there be no persons who can direct their wills to the same ends in one and the same world. Right then is only potentially right and not yet that which according to its conception it is to be, so long as it only denotes by anticipation the approval or disapproval of relations which do not yet exist.

Its 'other-being' is also quite intelligible; it all comes to the simple truth that general conceptions mean nothing when there are no particulars for them to connect; the 'other-being' of right consists in the various rights whose conditions lie in the existence of this nature, of these human personalities with these definite wants and claims; after the general doctrine which sets forth the conception of right will come the special doctrine which contains its applica-

tions. This direction is so simple that we do not need to wait for the dialectic method to teach it to us; but that method does not help us in the least to carry it out; for after all experience alone can teach us what conditions do in fact exist which give occasion for the development of the general idea of right into special forms of right.

194. There is, however, yet another kind of advance that we can conceive. 'Other-being' certainly does often mean the passing of the universal into its various particular forms; but I have already remarked that the Hegelian doctrine lays stress upon the relation of opposition which prevails between the two members, including the opposition of the universal to the particular: this idea of opposition, universalized and carried to its extreme pitch in the conception of contradiction, gives a further meaning to 'other-being,'—it may stand for the simple contrary of that which the first (the being in itself) stands for. In pursuance of this train of thought, right was made to pass into wrong; and wrong was made to issue in punishment, not indeed as the 'being-for itself,' but as the means of reasserting the violated right by the negation of its 'other-being,' i. e. of the crime.

Now here again we have nothing that would not be just as clear by itself without all this apparatus of the dialectic method; and further, the method is actually confusing. Any unprejudiced person would say to himself on reflexion that all right has living reality only when living persons not only know it but respect it in their actions, but that the movements of men's wills are not in fact governed by the ideal which they *ought* to follow; wrong and crime therefore appear, not as something necessary that *must* exist, but as something possible that *may*, and indeed always *will*, exist, to judge by what experience teaches us of human nature. In the transition which the dialectic method gives there is none of this cautious bridging of the gap between the two conceptions; it is represented

as part of the very conception of right that it shall pass over into wrong, and the paradox is not to be justified by a plea which will be presently considered.

The transition to punishment as the third stage offends us less merely because we supply the motives which are in truth not given at all by the method itself. The method does indeed demand restoration of the right, and that by negating its negation the wrong; but it does not tell us by what procedure this task, stated abstractly as the negation of the wrong, is to be carried out. Why should it take the shape of punishment? The evil disposition out of which the wrong sprang is equally negated by disapproval and by improvement, the harm done by payment of damages, the violation of the dignity of the law by repentance, and by a fresh recognition of its bindingness. All these considerations show that the dialectic method was of no use here except as a scheme, with places marked out which we might seek to fill, but that, though we were tolerably successful in filling them, the content with which they were to be filled was only to be got from a quite independent examination of the peculiar nature of the object in question.

195. We said that it seemed to us absurd to maintain that it is part of the very conception of right to pass over into wrong; but this *swinging round of a conception into its opposite* has been so often and so emphatically claimed as a higher truth discovered by dialectic, that it is worth while to return to the point. Hegel remarks[1] that at first of course the understanding fancies it can apprehend the nature and truth of the real world by a number of fixed conceptions complete in themselves and exclusive of each other; but that the truth is that different conceptions do not simply stand one beside the other with equal claims to represent the finite, but that the finite of its own nature does away with itself, and passes over of itself into its contrary. Thus

[1] [Vol. VI. of his collected works, p. 152 f.]

we say that man is mortal, regarding death as something whose ground lies merely in external circumstances; and according to this view man would have two distinct properties, that of living and that of being mortal also. But, according to Hegel, the true way of regarding the matter is that life as such contains the germ of death, and that in a word the finite in itself contradicts, and thereby does away with itself.

Here we can detect, more readily than we can in some of the other passages in which Hegel treats of dialectic, a confusion between two different statements. It is to the conceptions by which we try to apprehend reality that fixity and completeness are attributed in the first sentence: it is not the conceptions but the finite thing to which we apply them that is said to pass over into its contrary,—and in this latter statement lies all the truth that the passage contains, which truth is shown by what follows to have been uttered unintentionally or even contrary to the intention of the author. For when the finite as such does away with itself, it does so not because the general conceptions which apply to it have lost their definiteness and swung round into their contraries, but because it, the thing to which those conceptions are applied, as finite or as actual, is unable permanently to fulfil what is required of it by these conceptions, though each of them is true of it at one moment; through a defect in its nature it passes out of the province of one unchanged conception into the province of another which is equally unchanged. But the conceptions themselves do not alter their eternal meaning because it is only for one moment perhaps that they are a correct measure of the changeable objects to which they are applied.

The true view of the matter then cannot be that life as such bears in it the germ of death, and that the finite in general contradicts itself: it is rather the two parts of this statement that contradict each other. Life as such does not die, and the general conception of life obliges the living

thing to live, not to die; it is only the finite, mentioned in the second part of the statement, i.e. only particular living bodies that carry in them the germ of death. And even they do so not in virtue of the idea of life which is realised in them, but assuredly only by force of external circumstances, i.e. only because that combination of material elements through which alone life is manifested on the surface of this earth is unable to exhibit an undying example of life, though that would in no way contradict the idea of life,—whether this inability be regarded merely as a result of the laws of nature which are here in operation, or as part of a universal plan.

Similarly right never itself passes over into wrong, but sometimes the will of a living person which ought to embody it may, through want of judgment or through the impulse of passion, be led into wrong while striving to do right, and sometimes the law, which, men being what they are, could not be administered at all if it allowed exceptions, may do a wrong in a particular case involving complications for which no provision has been made.

Logic then can in no way accept this doctrine that conceptions dialectically do away with themselves: but the real world as we find it is so arranged and ordered that what is, though it does not do away with itself, yet does of its own nature pass from the province of one conception into that of another; and the fact that we find it so is worth notice, as a fact about things that is to say, not as a peculiarity of the intellectual tools by which we come to know them.

196. In any case, even apart from all the objections here raised, the dialectic method would in the end give us only an arrangement of our conceptions,—an arrangement which might no doubt present various points of interest to persons fond of reflecting and comparing, in the æsthetic impression produced by the discovery of analogies, parallels, and contrasts, but which would scarcely open up a new way of knowledge that could lead to definite new judgments or

propositions, or to a better and more precise settlement of questions hitherto doubtful. To supply this want which the dialectic method fails to supply is precisely the aim of other vast attempts, viz. the attempts to found a logical language, a *universal mode of characterising conceptions*, or a philosophical calculus, at which Leibnitz laboured so long. The mere addition of a series of large numbers would be an endless task if we were obliged to have a distinct image of each one of the thousands or hundreds of units composing them, and to build up each of these numbers separately and at last their sum by repeatedly adding unit to unit. But our system of ciphering enables us, without the need of distinctly forming even any collective idea of the numbers, to set units under units, tens under tens, hundreds under hundreds, and then, by adding up each of these simple columns, unerringly to bring out a result which itself in turn we are quite unable to represent adequately in a single picture by any effort of our imagination.

Now our conceptions so far resemble numbers that they also contain for the most part a variety of individual images, whose union with each other is not distinctly before us at every moment, but only thought of in one collective impression; but they are denoted by words far less perfectly than numbers are by figures. By the use of words that are akin (though we are often no longer conscious of the fact) speech does indicate the kinship between contents, but very imperfectly, for kindred ideas are also denoted by independent roots: the kind of kinship between them is no less imperfectly expressed, for the small variety of ways in which derivatives may be formed is quite inadequate to the manifold relations that have to be indicated; moreover, instances of each relation occur which, as the first to take the fancy of the framers of language, are denoted by simple words in which the characteristic derivative form is wanting; and finally the name of a conception never gives us all the ideas that make up its content marked by simple signs and

united in such a way that when we have to combine several conceptions *M N O* we may shut our eyes to the meaning of the whole and apply ourselves to combine some of the component ideas with the same certainty of arriving at new and correct results that our system of ciphering gives us in numerical calculations.

These defects of language then we are called upon to try to amend; we are to dissect all our conceptions till we have found the simple primitive ideas of various kinds which admit no further analysis and the simplest ways in which they can be combined, and we are to characterise these by fixed signs, in order to obtain by their combination a symbol for each conception which shall adequately express its content. We need not think that the object of this undertaking is the formation of a new speakable language, which could never supplant the national and historical forms of speech: its result would be a collection of formulae for the purposes of scientific thinking only, to which recourse might always be had for the settlement of the doubts which arise from the employment of ambiguous expressions: for Leibnitz flatters himself that if we once got such an instrument disputants would always cut their quarrel short by an amicable agreement—'Let us reckon it out.'

197. This is no doubt one of those enterprises whose execution alone can finally decide whether they are practicable; it would be over-hasty to deny the possibility of that which might after all perhaps be realised by a happy invention. However, the utter want of success hitherto makes the inherent difficulties of the task more evident for the present than the possibility of overcoming them. If all we had to do were to make a system of signs for marking the contents of our conceptions, the problem might appear difficult but not insoluble. For then we should probably begin by passing over all the generic conceptions of natural history and limit ourselves to those conceptions whose

union in thought leads to difficulties which impede science or the practical deliberations of life. Nevertheless even this problem is harder than it seems, and the possibility of solving it derives only an apparent confirmation from the mathematician's language of signs and the symbols of chemistry.

It is characteristic of the mathematician that he reckons only with comparable elements, with magnitudes, the simplest combinations of which he certainly can symbolise quite clearly and unambiguously; but as the functions and equations thus obtained grow more and more complex, we see more and more plainly even here a sort of deterioration in their employment. In the place of denominations which really exhibit the inner structure of the magnitude in question so as to indicate quite plainly how they are to be treated in the calculation, we find introduced in order to secure the necessary conciseness arbitrary symbols which no longer have this property, but resemble the words of ordinary language whose meaning must be known quite independently of their sound. The expression $\sqrt{-1}$ still expresses the origin of the function for which it stands, and from this we can determine by general rules what results when we multiply it once or several times by itself: but this expression has already been discarded as too lengthy and replaced by the other expression i which as it stands gives no clue to its signification, and whose meaning must be otherwise already known if it is to be used correctly. When we go on to speak of B-functions and Γ-functions, these expressions are certainly concise, but we can only understand them by representing them as equivalent to other lengthy formulae, which in turn are only made intelligible by a previous explanation of the meaning to be attached to the general signs of magnitude and symbols of combination employed in them. All this is no reproach to mathematics, nor is it any proof of the impossibility of a universal system for characterising conceptions; it only

shows that any formulae that the latter could give us would not by themselves tell us all we need know, but would presuppose a great deal which we should have to learn before we could even understand them.

The symbols of chemistry make this still plainer: as yet they refer only to the quantitative relations of the combining elements, and to some extent to the supposed form of their union; what letters are to stand for the several elements, and how their sequence is to denote the arrangement of those elements, we must of course learn or know by heart, as both can only be determined by convention: but no one can tell merely by looking at the formula thus constructed whether it stands for a gas or a fluid or a solid body, nor what its density is or its specific gravity, nor what its colour may be, whether it is fixed or volatile, soluble in water or insoluble. If a man after looking at the formula answers these questions correctly he does so upon the basis of analogies with which his experience supplies him, and which he could not draw from the formulae themselves with any certainty that they would be correct. And yet all that is wanted here would be the determination of properties or modes of relation, which though not absolutely homogeneous are yet as physical processes dependent upon one another and functions of one another, and therefore give room for hope that laws may be discovered which will make it easy to mark by signs their dependence upon each other: but the difficulties would be vastly increased when we tried to characterise all our conceptions and had to deal with the combination of unhomogeneous elements which yet have a necessary relation to each other.

198. But it is not a system of signs only that we want, nor is the success of mathematics due to its symbols, though the skill with which they have been chosen has no doubt furthered its advance. The truth is that the usefulness of the signs rests here upon the fact that we already have unambiguous *rules*, which enable us to deter-

mine what follows from the simplest combinations of magnitudes, and then being applied anew with the same freedom from ambiguity to the results thus obtained issue in these elegant and certain methods of solving problems. It is these rules that we must feel the want of when we try to combine conceptions which denote something more than magnitudes so as to produce a certain result; and I believe that we have absolutely no reason to flatter ourselves with the hope that these rules would of themselves suddenly become perfectly clear so soon as we had analysed into their ultimate constituents the essences, contents, and matter to which they were to be applied. Assuredly there is no need to insist on the fact that increased clearness in the objects cannot but have a favourable effect on the certainty of our conclusions regarding them; but in the main it is not by analysing our conceptions and tracing them back to primary conceptions, but by dissecting our *judgments* and tracing them back to simple *principles* that we must hope gradually to fix our convictions which on so many points are still in flux.

But there are two things which we shall require to know: first what are the necessary consequences which follow from certain definite relations which, as we either arbitrarily assume or are forced to believe, hold between the contents of various conceptions; and secondly what general laws, not proved to be necessary but found to hold good in fact, connect various ideas in such a way that our reason, founding upon these laws, can deduce the consequences that will then necessarily follow from given conditions. These problems, which concern the application of the form of judgment, we must for the present attempt to solve without the valuable assistance which that universal system of signs would no doubt afford if it were once completed.

Note on the Logical Calculus.

THE idea of a logical calculus has been often taken up and often abandoned: but the Englishman Boole has recently made an elaborate and careful attempt to carry it out, which is beginning to attract attention in Germany as well as in his own country. Though I freely admit that the author's ingenuity makes his able work[1] very charming, I am unable to convince myself that this calculus will help us to solve problems which defy the ordinary methods of logic.

Boole does indeed insist that the result of a calculation when completed must be expressible in logical terms; but he holds that between the statement of the problem and its solution a course of operations may be introduced whose several steps allow of no logical interpretation; and he appeals to the extension of mathematics by the introduction of imaginary quantities. This appeal is hardly relevant. The mathematician could not avoid imaginary formulae: he lit upon them in the course of well-founded calculations: he has always sought for the interpretation of the enigmatic expression and has actually found it in the province of geometry. In the logical calculus on the contrary this working in the dark to which recourse is had from time to time would have to take place by means of symbols which have been arbitrarily chosen to denote logical elements and the relations of these elements. If therefore a calculation is really of use only when it allows us to solve single problems mechanically, without requiring us to be conscious at every moment of the logical meaning

[1] ['An Investigation of the Laws of Thought,' London, 1854.]

of what had taken place, it becomes all the more necessary that the rules which make such labour-saving processes possible should be determined upon purely logical principles without any rash and misty analogy from the province of mathematics. Though on this point I entirely agree with the admirable exposition of Schröder[1], yet I cannot entirely follow him: his demonstrations, which after the manner employed by mathematicians follow upon the statement of the theorems to be proved, have in my opinion no significance beyond that of establishing that the whole calculus is consistent with itself and that all the transformations and combinations of its elements which it allows lead to the same results when applied to the same problems: but we can only feel confident that the calculus as a whole is applicable, when it has been directly shown that each universal proposition is only the transcription of a logical truth into the symbolic language that has been adopted.

It has long been the custom in the section of logic that deals with artificial classifications to make use of letters to denote the marks which combine in various ways to form the different species that fall under a concept. Supposing that the three marks A B C belonged to the general notion M, the principle of disjunction would direct us to reduce each of them to its subdivisions a_1 a_2 $a_3 \ldots$, b_1 b_2 $b_3 \ldots$; the complete set of triplets of the form $a\, b\, c$, of course not counting repetitions or permutations, would represent all the kinds of M, which, failing any closer determinations, may be regarded as equally possible. These groups obtained by combination express *per se* merely the simultaneous presence of their elements; they leave the nature of the connexion between the latter undetermined in two respects.

First of all they do not assign the final form which is to be the result of the completed combination. Where logical

[1] ['Der Operationskreis des Logikcalcüls,' Leipzig, 1877.]

classification is aimed at this want is supplied by the image which is retained in thought of the abstract *M*, of which the kinds are in question; this *M* is to be added in imagination to each combination *a b c*, as the general outline which the union of the elements is to fill in; apart from such an occasion for the procedure by combinations, *a b c* taken by itself only designates any object of thought, no matter how constituted, in which the marks *a*, *b*, and *c* are found together, or what is more important, any case, which it has not yet been possible to characterise more closely, in which the *conditions a*, *b*, and *c* are found together. This uncertainty does not exist in mathematics, for the form which the result of the calculation is finally to take, is here completely and solely determined by the definitely assignable nature of the connexion which this science requires to be introduced between its elements.

Now with regard to this second point also, the reciprocal determination of their component parts, the formulae employed in the combinations, in themselves, contain no explanation of any kind. In algebra custom has made them an expression of multiplication; the particular sign of this operation which has to be retained in the case of arithmetic has been found unnecessary, in the case of algebraic calculations at least, and the product of multinomials has been found equal to the sum of the combinations of their elements. Logic, on the other hand, does indeed presuppose every mark that belongs to a whole to be connected in a particular way with every other, but it has no means of actually expressing these specific determinations, and entrusts them to our independent knowledge of the subject. But what universal laws it does possess on its own account with regard to the connexion of the marks bear no resemblance to the idea of multiplication. I will not here lay much stress on the fact that the multiplier, which must be thought of to begin with as a whole number, leaves the value of the multiplicand as a separate number unaffected

and only repeats it several times over; while every mark c, which is annexed to a combination $a\,b$, not only modifies the reciprocal determination of these original elements, but at the same time by adding to the matter of the thought limits the extent of its application. Anyone who cared to dispute the question might perhaps find it easy even on this point to make more of the analogies between the two sets of relations than of their differences. But it is an essential fact for our purpose that while multiplication is forced to retain both the recurrences $a\,a$, $b\,b$, and the permutations $a\,b$, $b\,a$, as indispensable components of its product, logic can admit no meaning in the former and no distinction between the latter. Thus the nature of the case presented no occasion for departing from the neutral significance of combination-formulae which can have many kinds of meanings, and applying to them the mechanism of calculation, which has strictly speaking no suitability to them except as symbols of quantities that can be multiplied. It could only be ventured on in the hope that the more extended application of the calculus would compensate, by results which no other means could attain, for a cumbrousness inevitable at the outset, seeing that exceptional rules were necessary to bring such an inappropriate mode of calculation into harmony with its logical object-matter.

Every A, according to the law of Identity, must $= A$. Natural thought has no motive to determine such an A over again by a characteristic A, in the same way in which A would be determined by a second mark b. No doubt we speak of a human being as truly human, or emphatically of a man who is indeed a man; but we only employ such expressions where it is permissible to distinguish the conception M of an ideal from the conception μ of the particular facts from which the realisation of the ideal is expected. At bottom, therefore, we are not determining a single M by itself. The human being $M\,\mu$ that is thus pronounced to be truly human, corresponds to its determination M once

only and then completely, and just so in another aspect corresponds to its zoological conception μ once only and then completely; such a thought bears no resemblance to the attempt to determine quadrupeds over again by repeating the character 'quadruped.' Nothing but the machinery of the calculus can suggest the requirement that a should be determined by a as in multiplication; but then the formula $a\,a = a$ or $a^2 = a$ which is now introduced to restore logical truth, should at least abstain from professing to be a newly discovered fundamental law of thought, or indeed anything but a make-shift contrivance to correct an improper procedure. The determination of a by a is logically speaking an operation that cannot be performed; it is only because and in as far as, in the context of our thoughts, such a fruitless attempt does not result in cancelling the a on which it is made, that it is permissible to substitute a by itself for the a^2 to which the calculus would bring us; but by no means to treat this a^2 as existent, and pronounce it equal to a. The left side of this equation contains an insoluble problem; the right contains, not the solution, but what has to be acquiesced in because there is no solution.

This is no mere verbal dispute, as may be seen from some considerations which Boole subjoins. If we accept $a^2 = a$ for an equation, it is an easy step to the inferences $a^2 - a = 0$ or $a - a^2 = 0$; Boole resolves this last formula into $a\,(1 - a) = 0$. Now the law of excluded middle teaches us that everything that is thinkable is either a or not a; this truth is expressed by Boole, who indicates the totality of the thinkable by the symbol 1, by saying that not-a is what remains of this totality when we subtract a from it; so that $(1 - a)$ is the contradictory opposite of a. Now the meaning of giving the equation the form in which one side is zero can only be that the combination on its left side has no extension that falls under it, and cannot therefore occur at all. Thus the formula $a\,(1 - a) = 0$ becomes the expression of the law that nothing thinkable

can be at once a and not-a. We may be delighted with the plasticity of the calculus which furnishes such a graphic expression of a familiar truth; but we shall be the less prepared to admit the interpretation which Boole gives his formula on p. 50 of his work. It shows, he contends, that the law which is regarded as the highest principle of metaphysics is only a consequence of a law of thought which is really mathematical in form; that it is because this law finds expression in a quadratic equation that our divisions and classifications have to be performed by dichotomy; and that if the equation had been of the third order we should have been forced to proceed by trichotomy.

I am sure that I shall not be guilty of trichotomy in the sense of hair-splitting if I object to this extraordinary piece of argument. Boole himself mentions that from $a^2 = a$ we can further deduce $a^3 = a$, but he disposes of this cubic equation with the remark that two of the factors which it presupposes, $\pm (1 + x)$, are incapable of logical significance; and it was clearly the same reason that decided him at an earlier stage to attach his inferences not to $a^2 - a = 0$ but to $a - a^2 = 0$. This procedure implies an idea which is quite correct; among the numerous formulae which can be mathematically derived from the supposed logical principle $a^2 = a$ none have any meaning but those which express something that is of use in logic; the validity of the logical law does not depend on the shape of the formula; it is the value of the formula as a symbol that depends on its agreement with the import of the law. But the quadratic form itself and its interpretation are altogether a mere caprice. I shall not insist on the point that according to $a^2 = a$, a should have been at once substituted for a^2, which would have brought us back quite intelligibly to $a - a = 0$; for even if we believed it possible to retain a^2 as a real result of a practicable determination of a by a, and *as such* to equate it with a, still there was no sort of logical justification for resolving $a - a^2$ into $a(1 - a)$. In mathe-

matics, where we are speaking of magnitudes, the transformation is correct and in it 1 really means unity; but in logic the difference $a - a^2$ does not present the least motive for regarding it as the product of two factors. The 1, which is introduced in doing so, is not unity, which it would have to be if the resolution were to be mathematically correct, but is Boole's arbitrary though not inappropriate symbol for the totality of the thinkable; the truth that a and $1 - a$ taken together exhaust this totality must therefore be established to begin with, in order to so much as make the interpretation possible by help of which the formula is intended to yield it.

These chimeras have not found their way to Germany; but I have mentioned them at length because of their connexion with a general conception which does meet with some assent among us. We do not overlook the differences between arithmetical and logical computation; but there is an inclination to the idea of a more general mathematical calculus[1], for which this distinction of object-matter would be indifferent. And it is true that every single act of thought, apart from the logical import of its result, admits of many uniform repetitions, and the result admits of many connections and rearrangements; further, the notions of equality, inequality, and opposition have significance even where they do not relate to magnitudes; though what consequences they have in such cases must of course be determined for each sphere according to its peculiar nature. Still, when it has been determined, when, that is, it has been decided under the jurisdiction of logic, what result must be derived from the combined or separate occurrence of several acts of thought and their particular results; then the recurrences and inter-connexions of all these elements may be embraced under the same rules of union, severance, and arrangement which hold good of all that is recurrent and that has number. Only the laws which are specifically

[1] ['Eines noch allgemeineren mathematischen Algorithmus.']

logical and, like the law of excluded middle, govern the formation of the actual elements which are to enter into this new connexion, must stand on their own feet; and it is an idea as incorrect as it is confused to expect that they can be established by any mathematics however abstract which should still merit that name in contradistinction to Logic. On the contrary, all that such a science would have to teach would be the development of the simplest logical truths, which are uniformly true of the manifold and its combinations, whether those of what has number and is homogeneous, or those of what has mere relations and is heterogeneous. Many things may be proved by mere verbal deductions; and so it may be held an important task to reckon up these truths, in their abstract form apart from their applications; I think it rather tedious than indispensable.

As direct expressions of such extremely simple truth we at once think of the axioms, the separate introduction of which is hardly more than a matter of form. Obviously the logical calculus must agree that $a = a$, and that every a and b which are equal to a third thing c are equal to each other; only the definition of equality demands a few words. Logic uses a to indicate a general mark, a general class, or a general case; and is therefore able to accept the language of the calculus, according to which a is the symbol of a class, whose extent comprehends all individual things or cases of whatever nature which share the character a. These relations of extent are all that the calculus notices; it therefore sets down two class-symbols, a and b, for equal when they present to thought classes composed of identically the same individuals and are therefore only two names for the same class. In such a case a and b may be different in themselves, even if their extensions are fully coincident; thus equilateral and equiangular triangles, if nothing but their extension is considered, are of course merely two names for the same class; still in logic we could not pronounce the two conceptions equal as regards the contents

which they directly declare as their own meaning. It follows just as simply from those simplest truths that it is always possible to comprehend two acts of thought and their results in a sum $a+b$; that $a-b$ is also possible in logic if the necessary homogeneity is obtained by b being included in a; that the other combination $a\,b$, which collects the two characters into one idea, represents a new class-symbol with a defined extension; and finally, that where the problem put before us is only that of carrying out some uniform mode of connexion, no difference can be made by the order of the *summanda* or factors which we combine to make a sum or product.

These easy analogies between mathematical and logical reckoning are less deserving of mention than the differences which are derived from the specific nature of logical thought. I have already mentioned the equation $a^2=a$; and not less paradoxical is the form in which the law $a+a=a$ veils the logical truth that each universal conception exists once only, that therefore every logical assertion about what comes under such a notion is completely exhausted when it is once thoroughly admitted of the conception itself, and that no new truth can be obtained by repeating the same process on the same object. Just so the theorems $a+ab=a$ and $a\,(a+b)=a$ remind us that every assertion which is once granted to be universally true of a is also true of every species of a that is still further determined by any mark b, and that therefore the mention of $a\,b$ beside a remains ineffectual, in other words, the former is 'absorbed' by the latter. It is only the improper employment of the sign of equation that gives these theorems their appearance of peculiarity; all that they really say comes to this; wherever the mechanism of the calculus would naturally lead to the forms a^2, $a+a$, $a+a\,b$, these useless incidents of its method are to be replaced for logical purposes by a simple a.

More important is the extended use which the calculus makes of the law of excluded middle; for the principle of

Duality, which appears at this point as a new law of thought, conceals nothing more than this familiar law. If we use a' to designate the contradictory opposite of a, and 1 for the totality of the thinkable, then we have, really as equations, the formulae $a + a' = 1$, according to which all possible matter of thought is exhausted by a and not-a, and $a a' = 0$ which declares the impossibility of a union of a and not-a. No further proof is either possible or necessary, whether for these laws or for the remaining one that the negation of not-a brings us back simply to a and not to any third thing; they are logical truths which have no doubt received in those formulae a very clear and convenient expression.

The old Logic had its chapters about immediate inference, conversion, and contraposition of judgments, and endeavoured by help of this same law to pursue the content of an enunciated judgment into its relations to judgments not yet uttered. Boole in a more comprehensive spirit sets before himself the problem of developing the different and mutually exclusive divisions of the thinkable that may be formed by the affirmation and denial of the concepts, class-symbols, or elements of whatever kind united in a judgment. If x and y are the given elements, and x' and y' their contradictory opposites, then xy, xy', $x'y$ and $x'y'$ are evidently the four classes into which all that is thinkable must be divided; that is, the constituent parts of the complete division which Boole calls the expansion or development of the given relation between x and y. It is somewhat inconvenient that following mathematical tradition he designates that relation between x and y as a 'Function' of the two, $f(x, y)$; logically such an expression can mean nothing, unless it is understood as the definition or predicate of some M, for then all the constituents xy, xy', etc., might be deduced from the given connexion between x and y, and with them the coefficients which would indicate them as possible or impossible within the extension of M. Boole

however employs for the moment the independent function $f(x, y)$ in order to develop out of it in an equally general way the law of the formation of those coefficients. His original equation $x^2 = x$, as he can find for it only the two arithmetical analogies $0^2 = 0$ and $1^2 = 1$, induces him to make the assumption that the logical and the mathematical calculus would completely coincide if these two values were the only ones which any magnitude could assume; and conversely, he takes all mathematical operations to be permissible in logic, on condition that the class-symbols to which they are applied are treated as magnitudes which admit of these two values only. So taking $ax + bx'$ as the given function $f(x)$, and $f(1)$ and $f(0)$ as the two values which it assumes if we take $x = 1$ and $x = 0$ (x' always assuming the opposite values), it is shown that $f(x)$ may be obtained by the combination of the two values: $f(x) = f(1)\, x + f(0)\, x$. The same consideration leads, in the case in which the given function contains the two elements x and y, to the formula:

$$f(x, y) = f(1, 1)\, xy + f(1, 0)\, xy' + f(0, 1)\, x'y + f(0, 0)\, x'y',$$

in which the two bracketed values refer in their order to x and y respectively.

If any stress is to be laid on this scheme of the logical development of a function, it would have been easy to establish it in a less bizarre fashion. It must after all be borne in mind that the zero which denies every magnitude alike, so that for every m, $0 \,.\, m = 0$ invariably, and the unit which every magnitude contains as a silent factor, so that for every m, $1 \,.\, m = m$ invariably, are exceptional and not merely homogeneous with all other magnitudes even in arithmetic. Granted that they rank as magnitudes when considered by themselves, still in combination or multiplication with other magnitudes they have the general logical import of affirmation and negation. What was required in the above theorem was only this logical mean-

ing, valid indeed for arithmetic but not derived from it; it was therefore improper to give currency to the illusion that logic is indebted to the peculiar laws of arithmetic for the instruments with which it operates. I will take two examples to show what I mean.

First, if $M = ax + bx'$, the value of the right side will obviously be reproduced if we first suppress the first term and leave the second, then suppress the second term and leave the first, and finally add together the two that are left:

$$ax + bx' = ax + 0 . bx' + bx' + 0 . ax;$$

then the coefficients can of course be expressed by $f(1)$ and $f(0)$, and

$$ax + bx' = f(1)\, x + f(0)\, x'.$$

Again, let the function $f(x, y) = ax + by$ be given and its development with reference to the terms xy, xy', $x'y$ and $x'y'$ required; and further, to make sure of what we are speaking, let us regard $f(x, y)$ at the same time as a definite M, whose definition, or specification of extent, is contained in the right side of the equation.

Within this M, the combination xy is possible in three cases, being the ax's which are also y, the by's which are also x, and the ax's which are also by in full, or the by's which are also ax in full; for none of these combinations are expressly excluded by the right side of the equation. We should therefore get axy, bxy, $abxy$; but as logically speaking the ab are included besides both under a and under b, it is sufficient to exhibit $a + b$ as coefficients of xy; and of course $a + b = f(1, 1)$ is equal, that is, to the value of the right side for $x = 1, y = 1$. The second term of the development would contain xy'; the equation tells us that if we suppress by which can never be combined with y' there can occur within the compass of M no y' or not-y besides ax; consequently a is the coefficient of xy', and a of course $= f(1, 0)$. Just in the

same way it follows that within M there can be no other x' or not-x but by; consequently $bx'y$ is the third term, and b of course $=f(0, 1)$. Finally the equation tells us that the extent of M is entirely exhausted by ax and by, and contains nothing that is neither x nor y; hence 0 is the coefficient of $x'y'$, and it again $=f(0, 0)$.

Thus there is no doubt that the proposed formula of function-development can be justified from purely logical considerations, and I would attempt to establish this on more general grounds if I saw more clearly what is the purpose of the whole proceeding. The first examples which Boole gives can only be regarded as exercises. If clean beasts x are according to the Jewish law those which divide the hoof y and chew the cud z, and then the development tells us there are no clean beasts which divide the hoof but do not chew the cud, and none which chew the cud but do not divide the hoof; that again there are no clean beasts which do neither the one nor the other, and lastly there can be no beasts which do both and yet are not clean; I have my doubts of the frequency of the logical desire to go through these deductions of given fact; but if any one feels the want, it is beyond a doubt more easily satisfied without a calculus than with one. But there are two other problems which Boole hopes to solve by help of such use of formulae; first, if a number of elements are given in any combination, the equation which expresses this combination is to be solved with reference to any of its elements at pleasure; and then it is to be possible to eliminate any one from the equation, in order to display the relations of the rest to one another.

As regards the first problem, I can only regret that Boole abandons himself recklessly to his principle of permitting himself all operations of reckoning if only their result can be logically interpreted. From the given proposition 'All men y are mortal x,' he obtains by contraposition 'No man is not-mortal' $yx'=0$. Now as $x'+x=1$ and therefore $x'=1-x$,

we get $y\,(1-x)=0$ or $y-xy=0$, $xy=y$; then further $x=\frac{y}{y}$, and by development of $\frac{y}{y}$ we obtain $x=y+\frac{0}{0}(1-y)$ or $=y+\frac{0}{0}y'$; this he takes to mean, introducing the mathematical significance of the symbol $\frac{0}{0}$: 'mortal includes all men and an indefinite number of what is not man.' Results that could only be obtained in such unwarrantable ways would certainly form no extension of Logic. Moreover, in this case such arts were not even necessary. For not the contraposited form $y\,(1-x)=0$ but the original $y=x$ should have been employed, only with the precaution of providing x from the beginning with a particularising factor v, $y=vx$. The proposition 'All men are mortal' means simply this and nothing more in the world besides; it merely regards y as subordinate to an x within the compass of which there is something else as well. There is no possible meaning in finding over again by calculation precisely what was presupposed, and what is self-evident, that is, that x comprehends beside the $v\,x$ which are y a further indefinite number of kinds which are not y; that therefore $x=y+wy'$.

With respect to the process of elimination, I shall content myself with giving an example. Every logical equation can, by applying contraposition to the affirmative judgment which it expresses, be reduced to the form in which one side is zero; for the equation $xz=0$ simply means that no x is z. I pass over Boole's doctrine about the procedure of collecting all given single judgments or equations into one solitary resultant equation, and suppress the scruples which I feel as to the necessity or productiveness of such an operation.

It is granted then that the equation is to be presented in the following arrangement; $pab+qab'+ra'b+sa'b'=0$; then the product of the coefficients $p\,q\,r\,s$ equated to zero,

is assigned as the result of the simultaneous elimination of a and b. This is easily seen with the ordinary appliances of Logic. For logically this equation cannot have the value o unless each of its terms taken by itself = o. Further, $p\,a\,b = 0$ says that No $p\,a$ is b; but $q\,a\,b' = 0$ gives by contraposition All $q\,a$ are b, and so in Cesare, No $q\,a$ is $p\,a$, or, $p\,q\,a = 0$, and this again gives No $p\,q$ is a, or by contraposition, All $p\,q$ are a'. Again $r\,a'\,b = 0$ gives, No $r\,a'$ is b; but $s\,a'\,b' = 0$ gives by contraposition All $s\,a'$ are b; so we get in Çesare, No $s\,a'$ is $r\,a'$, or, $s\,r\,a' = 0$, or, No $r\,s$ is a'. If we subsume the second conclusion No $r\,s$ is a' under the first All $p\,q$ are a', there follows in the same figure, No $r\,s$ is $p\,q$ or $p\,q\,r\,s = 0$. It is easy to see that if a similarly arranged equation with one side zero contains besides a, b, and a', b', more such pairs of opposites c, c', the elimination may be continued in the same way. But no doubt for such cases there is value in the abbreviated rule that the result of the elimination consists in the equation of the product of the coefficients to zero. If the equation had contained besides a term $z = 0$ independent of the pairs to be eliminated, it would persist without change, and might be added to the preceding term, so that in the result $p\,q\,r\,s + z = 0$ each of the terms by itself is $= 0$. Schröder remarks on this question at p. 23 of his work that the results of the elimination of a symbol a from several isolated equations are less comprehensive than those of an elimination from the combined final equation; $x\,a + y\,a' = 0$ and $p\,a + q\,a' = 0$ when taken apart, only give $x\,y = 0$ and $p\,q = 0$; while on the other hand the combined equation gives $x\,y + q\,x + p\,y + p\,q = 0$; and for this reason he thinks the latter order of procedure preferable. Is he not in this artificially making little difficulties, simply out of the order of procedure, which must ultimately depend on the development of the functions? Why are we forced to unite the four terms $x\,a = 0$, $y\,a' = 0$, $p\,a = 0$, and $q\,a' = 0$, although they must be true by themselves, in two equations, instead of regard-

ing them as four terms to be employed at pleasure? Then we might find without difficulty all results of elimination which we had any interest in ascertaining.

I do not maintain that the same syllogistic process will easily bring us to our goal in every case, especially in more complicated cases. But Boole himself insists that we must carefully analyse what we mean in every case, before translating our notions into the language of the symbols; and I certainly believe that the fulfilment of this postulate would enable us to dispense altogether with the calculus, and that Logic would prove rich enough to allow of the invention of adequate means of solution corresponding to particular problems, even if these means were not stereotyped beforehand. With reference to this point I mention a problem which Boole[1] puts and which Schröder repeats.

It is assumed to be known from an analysis of experience that in a certain class of natural or artificial products the combinations of the marks *a b c d e* are subject to the following rules; and in such a way, that not only the occurrence but also the non-occurrence of each particular mark belongs to the conditions from which the presence or absence of the others has to be inferred.

1. Wherever *a* and *c* are absent at the same time, *e* is present, together with either *b* or *d*, but not with both;

2. Where *a* and *d* occur, but not *e*, *b* and *c* will either both be found or both be missing.

3. Wherever *a* is found in conjunction with either *b* or *e* or with both at once, either *c* or *d* will be found, but not both together.

4. Conversely, where, of the pair *c* and *d*, the one occurs without the other, *a* will be found in conjunction with either *e* or *b* or with both at once.

It is required to ascertain:—

1. What can be inferred from the presence of *a* with reference to *b*, *c*, and *d*;

[1] ['Investigation of Laws of Thought,' p. 146 ff.]

2. Whether any relations, and if any, what, exist between *b*, *c*, and *d*, independently of the other marks;

3. What follows from the presence of *b* with respect to *a*, *c*, and *d*, and

4. What follows for *a*, *c*, and *d* independently of the other marks.

Boole anticipates that no logician would find the right answers to these questions by syllogistic process, unless he knew them beforehand; I fully admit this, but who would be tempted to select that process for attacking this problem while the more suitable one offers itself spontaneously? We have only to make a list (it is a purely mechanical process) of all the combinations of five which can be formed out of $a\ b\ c\ d\ e$ and $a'\ b'\ c'\ d'\ e'$, avoiding repetitions and the inclusion of contradictory elements, and then, or while making the list, to suppress those which are excluded by the totality of the given conditions. This leaves only 11 combinations;

$a\,b\,c\,d'\,e$	$a\,b'\,c\,d'\,e$	$a'\,b\,c\,d\,e$	$a'\,b'\,c\,d\,e$
$a\,b\,c\,d'\,e'$	$a\,b'\,c'\,d\,e$	$a'\,b\,c\,d\,e'$	$a'\,b'\,c\,d\,e'$
$a\,b\,c'\,d\,e$	$a\,b'\,c'\,d'\,e'$	$a'\,b\,c'\,d'\,e$	

From these we can read off the answers to the questions proposed:

1[1]. We infer from the presence of *a* that either *c* or *d* is present, but not both, or else that *b*, *c*, and *d* are all wanting.

2. There is no independent relation between *b*, *c*, and *d*, for all conceivable combinations of them with b', c', d' are equally realised.

3. From the presence of *b* it follows that either *a*, *c*, and *d* are all absent, or some one alone of them is absent.

4. If *a* and *c* are both present or both absent, *d* is impossible.

Similar questions about *e* which are not proposed could

[1] [Cp. Boole, pp. 148–9.]

be answered out of the same conspectus without any distinct operation.

I borrow from Schröder's treatise for purposes of comparison no more than the beginning of the solution by calculation; not so much to show that if all the intermediate terms are actually supplied it is by no means distinguished by brevity, but chiefly with the general object of elucidating the use of the calculus by help of an instance that involves a real problem, and is not merely going back upon what we know to clothe it in awkward formulae.

By contraposition of the positive judgments which constitute the given conditions of the possible combinations, and so reducing them, as equations, to the form in which one side is zero, we obtain

from 1. $a' c' [e' + b d + b' d'] = 0$;
from 2. $a d [b c' + b' c] e' = 0$;
from 3. $a [b + e] [c d + c' d'] + [a d' + c' d] [a' + b' e'] = 0$.

As the questions ask nothing about e and e' the first operation to perform is the elimination, which we dispensed with, of this pair of opposites. According to the rule given above its result consists of equating with 0 the sum obtained by adding those components of the equations which are free from e and e' to the product of the coefficients of e and e'. Now to begin with, the coefficient of e in 3. $= a (c d + c' d')$, and that of e' in 1. 2. and 3. $= a' c' + a d [b c' + b' c] + b' [c d' + c' d]$; the product of the two is according to the above-mentioned rules $= a b' c d$, and with the addition of the terms free from e and e', which are $= a' c' [b d + b' d'] + a b [c d + c' d'] + a' [c d' + c' d]$ the entire result of the elimination would have to be brought together into

$$a [c d + b c' d'] + a' [c d' + c' d + b' c' d'] = 0.$$

Now to answer by this result in the first place the second question, about the relations between b, c, and d, we should have to eliminate a and a'; but the requisite product of their coefficient is $= 0$ because each individual product as

it arises takes independently the value o owing to the combination of contradictory elements; the result is therefore $0 = 0$, and we must accept this as a sign that there is no independent relation between these three marks. However, we see at once that if we give the symbol p to the coefficient of a that of a' will become p' or not-p; we therefore obtain from $a p + a' p' = 0$ the two equations $a p = 0$, or No a is p, and $a' p' = 0$, or No not-a is not-p. The first of these gives at once; all a are not-p, or p'; hence $a = c d' + c' d + b' c' d'$, which formula answers the first question.

I omit the continuation which would be needed to answer the third and fourth questions, and confine myself to remarking that in the whole of this problem no use has been made of the development of functions, of the importance of which I expressed my doubts above; the required equations were obtained directly from the given propositions, and the eliminations out of them were conducted on a method, the origin of which we explained to ourselves by help of syllogisms in the second figure. Thus there is nothing to be said against the appropriateness of the present method; but just as little against the superior simplicity and plainness of that which we adopted. This, by the way, had not to wait to be discovered by Jevons, for it was already forthcoming in the doctrine of classification, which long since required in the first place the tabulation of all the marks in their combinations, and then the cancelling of all combinations that become inadmissible on taking account of the reciprocal determinations of the marks. I cannot therefore convince myself of the advantages to be derived from the attempt to systematise in a fixed logical calculus all the means of vivid and abbreviated presentation to which every one has spontaneous recourse in given cases, applying them with variations adapted to the proposed problem. It is inevitable that a symbolic method intended to make uniform provision for every case should purchase

its suitability for the solution of one problem at the cost of a useless prolixity in its treatment of others and of manifold discords with the custom of language.

Even the quantification of the predicate, which was the starting-point of recent English logic, was no new discovery, but the superfluous inflation of a familiar idea to an excessive importance. That the predicate of a judgment, except in case of simply convertible judgments, has a larger extent than the subject which in part takes its place within this extent; that therefore it is not merely the predicate that determines the subject, but also the subject that restricts the predicate to such a modification as is true of the subject's self; these were old doctrines of logic, and in its rules of conversion it went so far as to provide for their application. It is true that the scheme of judgments gave no special expression to this truth, just as the ordinary linguistic form of the sentence did not. But what harm was there in that, when the fact was known? Did the want of such an expression ever deceive a considerate thinker? And was it worth while, for the sake of amending such trifles, to have recourse to such dangerous contrivances, as to connect the natural expression of thought with a new symbolism and a new calculus? There could be no real gain in expressing the proposition 'All men are mortal' by $y = v\,x$ unless a means could be discovered of defining this v; as long as it remains an undefined coefficient, it is an ineffectual indication of what we knew before. In the converse of this judgment 'Some mortal is man,' the old logic would bring to light this indefinite Particularity[1] neither better nor worse than that v would; if we object to the expression 'some,' our objection might be easily removed by the consideration that these indefinite particular judgments are at the same time forms of modality, and express the possibility of a conjunction of their predicate with the general notion which forms their subject, by

[1] ['Diese unbestimmte Particularität.']

affirming such a connexion for some but not for all cases of the notion.

There is a passage of Jevons ('Principles of Science,' London, 1877, p. 59) which among others has occasioned these remarks. He forms two premisses; sodium[1] = sodium metal, and sodium = sodium capable of floating on water. He draws the conclusion sodium metal = sodium capable of floating on water. To this he subjoins these remarks. "This is really a syllogism of the mood Darapti in the third figure, except that we obtain a conclusion of a more exact character than the old syllogism gives. From the premisses 'Sodium is a metal' and 'Sodium floats on water' Aristotle would have inferred that 'Some metals float on water.' But if enquiry were made what the 'some metals' are, the answer would certainly be 'Sodium[2].' Hence Aristotle's conclusion simply leaves out some of the information afforded in the premisses; it even leaves us open to interpret the 'some metals' in a wider sense than we are warranted in doing. From these distinct defects of the old syllogism the process of substitution is free and the new process only incurs the possible objection of being tediously minute and accurate." Oh no! we might admit the 'tediously,' but otherwise Aristotle is in the right. Jevons' whole procedure is simply a repetition or at the outside an addition of his two premisses; thus it merely adheres to the given facts, and such a process has never been taken for a *Syllogism*, which always means a movement of thought that uses what is given for the purpose of advancing beyond it. So the combination of words which he proposes is not a syllogism at all, and consequently not one in Darapti. The meaning of the syllogism, as Aristotle framed it, would in this case be that the occurrence of a floating metal Sodium proves that the property of being so light is not incompatible with the character of metal in general. If he

[1] [See Professor Lotze's Preface to the Logic.]
[2] ['Metal which is Sodium,' Jevons.]

expressed this by saying 'Some metal is capable of floating,' he intended of course not to repeat the premisses which were known before; but to enunciate the possibility of a general distribution of this property among metals, as a supposition whose correctness in fact there is ground for testing further, since it is logically not inconceivable. Even the expression 'Some metal' is at bottom quite correct, for Sodium certainly is some metal; the expression does not enjoin us to think of other metals at the same time with it; it is true that it does not prohibit our doing so, but this need not give rise to any error.

How often have modern enterprises like these proclaimed the dawn of a wholly new epoch in logic, and the fall of the contemptible system of antiquity! I am convinced that if the ancient logic were to be really forgotten for some generations and then rediscovered by some fortunate thinker, it would be welcomed as a late discovery, after long search, of the natural march of thought, in the light of which we should find intelligible both the singularities and the real though limited relevancy of the forms of logical calculus with which we had made shift so far.

CHAPTER IV.

The forms of Proof

199. It was our business in writing of systematic logic to enumerate the various forms of judgments and to point out the precise mode of union which in each of these forms is conceived as subsisting between S and P or as to be effected between them: it is the business of applied logic to consider what contents S and P can properly be joined in one of these forms of union. Various problems which we shall not always hold apart fall within this scope. In the first place the communication of the thoughts of others gives us numerous propositions of the form S is P, whose meaning and purport is perfectly plain, but whose validity is questionable: then there arises for us the problem of a *proof* for the *given* proposition T. In the second place our own observations may lead us to suppose that between two ideas S and P there subsists a relation which if it were known could be expressed in a judgment of the form S is P: then we are called upon to *discover* the yet unknown proposition T which would be the precise expression for this supposed relation.

These two, discovery and proof, differ only in their different use of the same materials. The same combinations of thought by which the truth or probability of a proposition T was first discovered may always be applied, when put somewhat differently, and sometimes even with-

out any such transformation, to prove the truth or probability of a given proposition *T*. Moreover it is easy to see that the reflexion of the discoverer, if it is not to miss its aim, needs at every step slight connecting links, resembling a proof in form: and conversely that a proof will never reach its goal without some inventive play of thought. On the whole however discovery reaches farther than proof: and so I will separate the two problems, though I shall not always avoid the natural mixture of the two. Scientific investigations lead to both in about equal measure; the needs of life more frequently lead to discovery.

I find reason however again to divide the first part of the subject, and to separate the proof of universal propositions from the proof of particular or singular propositions. It is true that a universal relation can seldom be established between *S* and *P* without the employment of knowledge supplied by experience; but as such knowledge, if it is to lead to universal conclusions, must itself have universal validity, we may regard it as knowledge which, though originally derived from our experiences, is yet, now that we have full confidence in its universality, to be counted among the proper instruments of thought. The proof of particular facts on the other hand, of historical events or of the ordinary transactions of life, can never follow from universal propositions alone, not even from such universal propositions as are themselves derived from experience: it presupposes the knowledge of a number of particular circumstances, occurring only here and only here united in this precise manner. The preliminary process of getting at all these conditions, from which the conclusion is to be drawn, requires peculiar instruments which we shall consider presently. The solution of a proposed problem on the other hand, even when the result is to be not a universal proposition, but the establishment of a single fact, may be connected with the proof of universal propositions: under the conditions which here do not need to be sought but

are given, and so far as they are given, the definite proposition T which satisfies them all is always to be found by employing instruments of thought which are of universal application, and these theoretical results are inaccurate and need correction in practice only so far as we have failed to state *all* the conditions which T had to satisfy.

200. Every proof is a syllogism, or a chain of syllogisms, which completes the premises required for the given proposition T, so that they fit into one another in such a way that T follows as their necessary consequence. But the validity of every conclusion depends upon the validity of its premises: these again might be established by fresh proofs, but this procedure would go on *ad infinitum* without any result were there not a number of universal propositions which we accept as immediate truths, which therefore neither need nor are capable of proof, but are themselves the ultimate grounds by appeal to which we may decide in every case whether a conclusion is correctly or incorrectly drawn from its premises. I do not intend as yet to discuss the question of the source from which we obtain these immediate truths: we are here concerned only with the mark which justifies us in classing a proposition T among the *axioms*, assent to which we believe ourselves entitled to demand from every sane person. Now it is conceivable that, just because there is no possible proof of an axiom, this mark may in the last resort be nothing but the *self-evidence*, the immediate clearness and certainty with which the content of a universal proposition thrusts itself upon us as a necessity of thought; and in fact this is what we always come back to in the end.

Experience however abundantly shows that propositions which later generations have proved to be false, were as self-evident to earlier generations and produced in them as strong a conviction as any propositions whatsoever: relations which in the limited sphere to which our observation is confined are seen to be permanently present or constantly

recurring, without any contrary experience to disturb us, very commonly assume the appearance of necessities of thought. There is only one way of distinguishing the spurious self-evidence of a prejudice from the genuine self-evidence of a true axiom: we must try whether the contradictory of *T* the proposition in question is as impossible in thought as *T* itself seems to be necessary. This test will often be quite decisive; we shall often find to our astonishment that the attempt to join *S* and *P* in the opposite way to that asserted by the given proposition *T* leads to no inner contradiction in our thought at all. In that case *T* is no axiom, but either altogether an error, or a truth that holds true in some cases only, or a truth which though universally true requires to be proved. In the other case, when the contradictory proposition non-*T* appears as impossible in thought as *T* appears necessary, we may with greater confidence regard *T* as an immediate axiom; but the test does not even now give perfect security, for it is quite possible that the inconceivability of non-*T* and the apparent necessity of *T* may both alike rest upon a spurious self-evidence. Should these two simultaneous errors be made, logic furnishes no short way of detecting them: our mistake could only be gradually amended by our becoming aware of the contradictions which experience offers to the assumed validity of *T*, and by a slow and far-reaching modification of our system of thought suggested by those contradictions.

Such a double error will seldom be found in the case of purely theoretical principles, more often in the case of the principles upon which our moral judgments are based, and which may be classed as genuine or spurious axioms, although strictly speaking they do not seem to be necessities of thought but only unquestionable truths, and their opposites do not seem to be unthinkable but only absurd. That you ought to hurt your enemies was for a long time generally accepted by the ancients as an unquestionable

maxim, and the opposite of it regarded as absurd: such errors can generally be removed only by a gradual alteration in men's habitual feelings.

201. Supposing now that T is a universal proposition whose validity is not axiomatic, i.e. that it is such as to need proof, we yet shall not set about proving it till we know that T is worth proving. In three cases it will not be worth proving. In the first place it will not be worth proving if its content is an incomplete, and therefore an indefinite thought. A man of untrained intellect, so long as he confines himself to the objects which naturally come within his scope, is usually conscientious in enumerating and examining all the points which are important for the understanding of a fact: he follows the old rule which tells us to ask 'quis? quid? ubi? quibus auxiliis? cur? quomodo? quando?' and to omit none of all these questions. But he is quite helpless when he wanders off into general considerations which belong to the province of speculation: he then usually does not get beyond a clumsy expression for something which he perhaps rightly believes, demands, or assumes, but is unable to connect with any determinate or determinable points. The philosopher on the other hand, revelling in his abstractions, does not always come to meet him half-way, but often contents himself with employing conceptions which when severed from their proper application become utterly meaningless: the result is that vague theses are nowhere so common as in the attempts of a man who has had no logical training to philosophise by the light of nature. That God and the world are one is a proposition that no one can prove except him who propounds it; so far as his proof is correct at all it is the proof itself that tells us what he meant by the proposition: any other person than he who propounded it will, if he be wise, attempt neither to prove it nor to refute it; for that God and the world are in some sense two is asserted by the proposition itself, for otherwise it could not have distinguished them; but that

they are also one in some one of the many senses of unity, may be supposed without more ado.

That things are appearances is an equally ambiguous proposition : the things which appear to our senses are so of course, for otherwise they could not appear to us : but that the things which though themselves inaccessible to observation we suppose to underlie our sensuous perception are also appearances is an incomplete thought till we determine what is to appear and to whom it is to appear. All these and other similar propositions are not worth proof or refutation, but are simply to be returned as they are to him who brought them, just as in a court of law we refuse to listen to a man who complains that he has suffered wrong without saying what has been done to him and who has done it.

202. The second case is when though a perfectly clear nominal definition may be given of S the subject, or P the predicate of the proposition T, the definition contains a combination of ideas which can be shown to be impossible, or cannot be shown to be real. No one would trouble himself to prove or to refute a proposition the subject of which is a wooden iron : no one would seriously enquire whether this wooden iron will burn in the fire like wood or melt in it like iron. There is no such logical contradiction in the ideas of ghosts and will-o'-the-wisps, but we defer asking whether the former need sleep, and whether the latter are attracted by buried metal, till their existence is proved. What we here require may be called in general the *justification* of a conception, which must without fail be added to its nominal definition when use is to be made of it.

This may be effected in various ways. If M stands for something which is supposed to have external existence, the shortest way to justify M is to *point* at once to an instance of it or to a fact in which its existence is given and accessible to observation. If M denotes a combination of ideas the validity of which means that it can be carried out and that its result can be imagined or realised in a

mental picture, this very realisation of the content which *M* demands, or in other words its *construction* will justify *M* itself: thus geometry establishes the admissibility of the conceptions it has formed by presenting in a visible form what they till then only contained as a problem, thereby proving most conclusively that the problem was soluble. If we can neither point out any instance of *M* nor carry out its construction, we must at least show cause or give a 'deduction' which explains how in connexion with some demonstrable reality or in pursuit of some problem we have been properly and justly led to form this conception. Such a 'deduction' cannot always directly prove the validity of *M* in the shape in which the conception is presented, but it may always show that *M* is a preliminary designation for some content which we are reasonably and rightly looking for; it remains for the further enquiry whose beginning is hereby justified to determine whether *M* itself can be justified as a valid conception, or else how its content must be modified in order to make it valid.

The ancients regarded the doubling of the cube as a serious problem: but though they could not geometrically construct the required line, whose cube should be double of a given cube, yet it was all along certain that the problem was soluble and that the required line was a magnitude which could in some way be found. For it could be shown that as the side of a cube increases its volume must also continuously increase without any alteration in its shape: among this infinite series of larger and larger cubes then must be found that particular one which is double of a given cube, and this implies that its side actually occurs in the series of existing lines. We here show cause for the necessary validity of that which is sought instead of actually realising it in a construction.

Again it may be doubted whether one and the same conception of length fits both curved and straight lines; but setting this doubt aside it was not unreasonable as things

then were to hope to find by a simple geometrical construction the straight line which is equal to the circumference of a circle of given radius; for it was certain that the length in question depends upon the length of this radius and upon nothing else. This hope was only banished by the completion of the enquiry, which showed that the circumference cannot be expressed as a determinate real and algebraical function of the radius. In the natural sciences a hypothesis often assumes facts which we can never hope to establish by direct observation: often indeed we must leave it to God and the future to show even the possibility and constructibility of that which we are for the present absolutely obliged to assume. The only way of justifying ourselves in such a case is to show from the given facts the pressing need of the idea which we employ, with the reservation of course that we may at a future time so alter it as to enable us to construct it without impairing its usefulness. We shall return to this point on another occasion; for the present it is enough to refer to the instances above employed as showing what kind of justification is needed for conceptions if their union in a proposition is to deserve proof or refutation.

203. We will now suppose that the conceptions which are joined in the universal proposition T have the requisite definiteness and validity: but even so we do not start in search of a proof that shall exhibit T as the necessary consequences of premises that must be discovered, until we have got some preliminary warrant that the proposition is true as a matter of fact; for it would be lost labour to try to prove what is not even true. If T is a universal proposition of whose field it is not easy to take a comprehensive survey, we first try whether T holds good in some examples that lie near at hand: a single case in which it did not hold good would do away with the universal validity of T, and the problem would then be changed into finding the conditions under which T has at least a

partial validity; if on the other hand that which T asserts is found to hold good in all the cases of its application which we compare, this trial, though being necessarily incomplete it cannot prove that T is universally valid, may yet corroborate what it alleges so strongly that it will be worth while to search for a proof. This very needful preliminary procedure, which will further on take its place among the forms of proof, is in fact neglected but seldom, and that mostly in cases where the validity of T cannot be tested by mere reflexion upon instances supplied by the memory, but only by observation or experiment. The courtiers of Louis XIII exhausted themselves in ingenious proofs of the proposition that a living fish thrown into a bowl full of water makes it overflow while a dead one does not, until the gardener who was called in made the experiment and showed the assertion to be entirely false; but others make the same mistake, and in the less exact departments of natural science we frequently find subtle demonstrations and explanations of phenomena whose actual occurrence is entirely problematical.

204. Supposing now that this preliminary question is settled, and that T is recognised as a universal proposition that deserves proof, its truth or falsehood may be established either in a direct or in a roundabout way, and this makes the first division of proofs.

A proof is *direct* when it shows immediately that the given proposition T is necessary or impossible; it is *indirect* (or *apagogic*) when it establishes the truth or the falsity of T mediately by showing the falsity or the truth of its contradictory non-T. In each case there are two directions which the train of thought may take. We may call a proof *straightforward* or *progressive* when it starts with that which in the nature of the thing conditions something else and makes that which is conditioned issue from it as its consequence; it is a *backward* or a *retrogressive* proof when it starts from that which in the nature of the thing is con-

ditioned in order to arrive at knowledge of that which conditions it. The first form of proof, since it proceeds *a principio ad principiatum*, may equally well be called *deductive*, though the opposite name *inductive* will not be found so generally suitable for proofs of the second form which proceed *a principiato ad principium*. And finally there is yet another distinction applicable to both these lines of proof: you may go forward (progressively) either from general truths to T or from T to its proper consequences, and similarly you may go backward (retrogressively) either from T's consequences to T, or from T itself to the truths upon which it is founded. We cannot pronounce upon the comparative value of the eight different forms thus obtained till we can consider each in reference to the problems for which it is usually employed. The following survey may enable us to do this.

205. The *first* form of proof, which is direct and progressive, proceeds from a universal truth, which is placed as major premise at the head of the whole procedure; in the minor premise (or in a series of epi-syllogisms, if the proof can only be completed in a chain of reasoning) it is then shown in what relation S and P which are joined in the given proposition T stand to that major premise; and lastly the conclusion infers that by reason of these relations of S and P the proposition T which was to be proved must hold good. If the problem be stated in this general way it seems as if all the three figures of Aristotle might be employed in this form of proof: the fact is however that the first figure alone answers to the spirit of it. I do not reject the third figure on the ground that as usually described it only gives particular conclusions, while we here wish to prove universal propositions; if we put the particular conclusion 'some S are P' into a modal form, 'that which is S may be P,' we get a universal proposition which it may be worth while to prove. For instance if we want to produce an effect P, and have nothing to get it out of except an

unpromising material *S*, we shall be glad to see it shown by a syllogism in *Bramantip* that *S* and *P* are compatible with one another in the case of a subject *M*, and that therefore *S* does not always make the desired effect *P* impossible. But the third figure does not exhibit this proof in the progressive form. It only states in the premises an instance of the coexistence of *S* and *P*, from which we may argue regressively, *ab esse ad posse*, to their compatibility. The second figure admits universal conclusions indeed, but only negative ones; these too may be valuable, but they cannot be obtained by this figure without premises of opposite quality, and therefore fail to satisfy us. For a universal negative proposition *T*, which simply denies a predicate *P* of a subject *S* because *S* and *P* stand in opposite relations to a third *M*, appeals to a *mark* which shows that *S* and *P* cannot be combined, but not to a *reason* which explains why they cannot: it merely expresses a fact which is indeed true, but remains unintelligible till we have learned in an affirmative proposition what *S* really is, and thus now can see that because it is this it cannot be the other, viz. *P*. And so the second figure, since it establishes its conclusions by proofs which, though appropriate and convincing, give no explanation, is also rather regressive than progressive in character. And therefore under the head of direct progressive proofs attention has usually been directed to the first figure, especially to its affirmative moods, and for the present purpose to *Barbara* exclusively: it is only here that we find the *subordination* of a given idea under a general truth, which enables us to understand not only *that* *T* holds good, but *why* it holds good.

206. This opinion is as old as Aristotle: it is worth while to observe however that this form of proof is to be regarded as an ideal in another sense than this: it cannot fairly claim the praise bestowed upon it unless we succeed in filling it with the content which its articulation requires, i.e. unless we set down for major premise a general proposi-

tion under which the special case of the minor premise demands to be placed in virtue of its very nature, and which therefore would actually be the reason upon which the validity of the proposition to be proved depends not merely for our reflexion but in the nature of things. But it is clear that we may use the form of this proof without in the least satisfying this last condition. Many instances occur, and that precisely in the field of mathematics where exact treatment is required, of propositions that admit of various equally convincing proofs all couched in this form of subsumption, none of which therefore can claim exclusively to express the proper connexion and development of the thing itself. The possibility of presenting the same idea in very various forms without altering its value enables us here to subsume it under a great variety of universal major premises, and to proceed from any one of these arbitrarily chosen starting-points to the same assertion *T*. I am anxious not to be misunderstood here and will therefore go into detail.

I will in the first place allow that we often find in mathematics a proposition *T* which is so evidently only an application of a definite major premise *M* that its deduction from this major alone seems natural, from any other artificial. I will remark in the second place that when *T* may be deduced with equal ease from a variety of majors *M N O*, I do not find in this alone any reason for saying that these various proofs are foreign to the natural sequence; for it may be (though I do not propound this as the true theory but only suggest it as a possible view) that the whole of our knowledge (e.g. of geometry) rests in fact upon a number of original and equally self-evident perceptions, none of which can be deduced from any other, but which, like the several components of one *complete* thought, are each and all valid at once and connected in definite ways with one another. We could then understand how in virtue of this connexion the same proposition admits of a variety of equally convincing proofs, according as we start

from one or the other of those inseparably united perceptions: no one of these proofs will exclusively exhibit the nature of the thing, but yet each may actually exhibit it in the form in which it is seen from that particular point of view; the possibility of a variety of proofs rests in this case upon the organisation of the content itself, which makes a harmoniously articulated whole not on one line only but on several lines at once. But I must nevertheless add in the third place that there remain many propositions *T*, whose proof (I mean in this form of subsumption) can only be effected by devices, which can be justified after they have been applied, but to the application of which we cannot find any invitation in the thing in question. It is to these proofs, of which many occur in pure mathematics, and a far greater number in applied mathematics, that the remark above made is intended to apply; though these proofs are as conclusive as can be wished, it is yet quite beyond our power to take them all in at one view, especially when they form chains of many links; and as they scarcely allow us to do more than see the necessary consequence of coupling each link to the one which follows, while the inventive ingenuity which forges the chain seems to be guided by pure caprice, we cannot honestly say that they *show* why the conclusion *T* is true; they only constrain us to admit that it is true. I have introduced this point because of its practical importance. Our ideal of knowledge and demonstration no doubt is that we should deduce each given proposition *T* from the determining grounds by which it is in fact determined *in such a way as to explain it*, and not simply assure ourselves of its *certainty* by a logical device; and if this problem is to be solved, it can only be by a direct progressive proof of this form. But it is soluble only within narrow limits, and where it is not soluble, where therefore we must content ourselves with the mere certainty of *T*, this form of proof by subsumption has not the least advantage over other forms. It is mere pedantry on the

part of the logician to wish in spite of this to enforce it and when a proposition can be conclusively proved in two words by an indirect method to look about for a direct deduction, which can only be effected by a chain of arbitrarily selected links, which makes it a longer business to get to that certainty, and which does not in the least help us to see the reason why it is so.

207. A *second* directly progressive method of proof is to start from the given proposition T, assuming it to be valid, and proceed to develop its necessary consequences. If among these consequences we find even one which contradicts either established facts or recognised general truths, T does not hold good as a universal proposition, and the proof becomes a mode of refuting a given proposition; it then includes, as may easily be seen, that preliminary procedure above mentioned, by which we assure ourselves before entering upon the actual proof that among the given cases there is no *contradictory instance* against the validity of the proposition to be proved. If the development of the consequences of T however far it be carried discloses nothing inconsistent with known facts or truths, we have not even yet got enough to establish the truth of T, for the next step in that development beyond the point at which we have stopped, might reveal the existence of a contradiction hitherto concealed, but at any rate this procedure suffices in the field of science to *recommend* a hypothesis, which is then reserved for further examination. But the true province of this method lies in practical life: it is the method we employ to recommend proposals, arrangements that are to be adjusted, resolutions that are to be adopted. And here the incompleteness of the development of the consequences is no obstacle; in all human affairs it is enough to ascertain what effects will follow from the application of a proposed measure within such a limited time and in such a limited field as we can readily survey: he who wishes to take count of all the subsidiary effects which

a microscopical examination might disclose, all the consequences centuries hence of what we do to-day, is a supercilious pedant; fresh measures will be taken to avoid minor disadvantages, and the remote future must take care of itself.

208. A *third* form, the first *directly regressive* form of proof, proceeds from the assumed validity of *T* and works back to the conditions under which this validity is possible. The difference between this form and that just discussed is not considerable, but there is a difference: it is not considerable because the conditions requisite for the validity of *T* can only be found by taking *T* as their basis and deducing them as consequences from it, a procedure which coincides with our previous direct progressive method: but we see that there is a difference when we consider the nature of that which is thus deduced. We may take as an instance of both forms at once the ordinary way of solving a problem in mathematics; for every such solution is at the same time a proof of the solubility of the problem, i.e. of the validity of the combination of ideas contained by the proposed problem *T*. If we assume that *T* is true and develop the consequences which flow from it, these consequences may be of various kinds; some of them will be particular circumstances which agree or disagree with given facts, others will be general relations between various objects which are either consistent or inconsistent with truths otherwise established. If we only come upon particular consequences which disagree with given facts or secondary conditions, we may with certainty infer that *T* does not hold, though we do not see the reason why it does not; if *T* is a practical proposal, it may be that it is quite acceptable in itself and that it is only its execution that encounters some obstacle; and then we should have a case of the second form of proof: if on the other hand we come upon absurd general propositions which must be true if *T* is to be true, then besides the certainty that *T* is im-

possible we get also a strong hint as to the reason why it is impossible; that reason must lie in the general truths which conflict with the absurd conditions we deduced, and herein we find what this third form of proof does for us. It not only clears the ground for the subsequent discovery of a direct and progressive proof of the contrary proposition, but gives us a remarkably conclusive and palpable negation of a given proposition *T* in the disclosure of all the absurd assumptions that would be necessary if it were true; and on this account this regressive proof is often preferable to a progressive one.

It cannot establish anything but the falsity of *T*, and so remains a form of *refutation*. If in working backwards from *T* we come upon none but admissible conditions, we cannot infer that *T* is true except in mathematics; for only in mathematics is it possible to develop from a proposed problem *all* the conditions necessary to its solution; in other cases we can never be certain that we have really deduced from *T* everything without exception that is implied as a condition necessary to its truth; the next step we took might bring to light an absurdity that we should have to assume. Affirmatively then this method is in matters of theory only able to establish the probability of *T*; in practice however we use it to recommend a proposal just as much as the foregoing progressive method. For when we want to secure the acceptance of a proposal we not only point out the consequences to be expected, but also show that the conditions of its execution are not incompatible either with the general requirements of justice and morality, or with the means which are actually at our command. A political measure always needs to be justified in these two ways,—after the former method by its useful consequences, after this method by the admissibility, in the view of justice and morality, of all that it implies: and in our daily life we must take count not only of the advantage to be expected from a provision, but also of the price we must pay for it.

209. A *fourth* method, the second *direct regressive* method, starts from given propositions and proceeds to prove from them the validity of T as the condition of which they are the result. This is a line of thought which we are very constantly impelled to follow: for the greater part of our knowledge of general laws is won in this way by reasoning back from given facts to that which must be assumed as the condition of their possibility. It is easy to see however that its most important applications belong to the method of discovery which tries to elicit from that which is given a T which is as yet unknown. When the general proposition T is given and we are looking about for the several propositions which may serve to confirm it, the proper method is always to begin with the progressive development of that which as consequence of T must be true if T be true: only when we have made a comprehensive survey of these consequences do we proceed to compare the result obtained with experience or with other truths, in order to reason regressively from the truth of this result to the truth of T.

I will therefore postpone the consideration of much that might be introduced here, and will only mention one species of this method, viz. that which infers the universal truth of T from its truth in particular instances,—*complete induction* or the *collective* proof. We are often compelled to employ it: it is not always possible to prove at one stroke that a proposition T holds good for all quantities, integral and fractional, positive and negative, rational and irrational, real and imaginary magnitudes; but each of these several kinds of quantities may offer some special point of attachment for a proof that T is true of it; if then we are sure that we have included all possible cases of T, that is in this case if we are sure that there is no conceivable kind of quantity besides those named, then we know that T is true of all quantities whatsoever. The general conception of quantity will then no doubt contain some reason for this universal validity;

nevertheless we cannot always point out this reason, or at least we cannot always make it quite clear and self-evident; and then we must have recourse to the collective proof.

210. The necessity of including without any omission all the kinds of cases to which T can apply if T is to be proved universally true leads here to an interesting special form of this proof. Mere completeness of course can always be secured by dividing all the cases into say Q and non-Q, the non-Q again into R and non-R, and so on as far as we like, stopping say at U and non-U: but this is seldom of any use; for even if we easily find separate proofs for the positive kinds of cases $Q\ R\ U$, it is very difficult to find one for the negative remainder non-U which embraces a miscellaneous crowd of different cases. We are constrained therefore to take a case Q, for which we happen to be already able to prove that T is true, and try to derive the other cases $R\ U$. ., etc. from Q in such a way that it may be evident that the changes by which Q passes into R, and R into U, either do not affect the conditions which made T true in the case of Q, or else constantly reproduce them. This is the method, familiar to mathematicians, first formulated by *Jacob Bernoulli*, of proceeding from n to $n + 1$,—chiefly applicable when the several cases in all of which T is to be true form of themselves a series in which each successive $(n + 1)^{th}$ member is formed in the same precisely definable way out of the preceding n^{th} member. If then it follows from the way in which the member $n + 1$ is formed from the member n that T when true of the latter must be true of the former also, it follows for the same reason that it must be true of the member $n+2$, and so on for every member of the series. For instance in teaching the elements of algebra this method is usually employed to prove the binomial theorem for integral exponents in a palpable way by repeatedly multiplying the binomial into itself.

The general idea of this proof however is by no means confined to mathematics, but is very often applied in com-

mon life, sometimes under the not quite appropriate name of a proof by *analogy*. In support of a plan or a statement we first mention an instance in which the plan is evidently advantageous, the statement obviously true; then we show that the other conceivable cases are in reality distinguished from this case by no feature that could possibly make a change in this respect; and thence we conclude that T holds good universally. It is easy to see how a careless or sophistical use of this method may lead to error. Between two very different cases A and Z we insert a great number of intermediate cases, each separated from the next by an inconsiderable difference d. Then instead of showing that if T is true of A it must also be true of $A+d$, which is B, we assume that it is so because d is so trifling; we reason similarly from B to C, and finally transfer the validity of T from A for which it really held good to a Z which by the accumulation of the many disregarded differences d has become entirely unlike A and does not in the least belong to the field to which T actually applies.

211. The indirect methods of proof may be treated more briefly. They bear formally the same relation to non-T that the direct methods bear to T, and the only circumstance that makes them in some degree peculiar is that we wish to arrive by them not at non-T but at T: they are therefore not affirmative but negative proofs in respect of non-T. The *fifth* method of proof, the first *indirect progressive* method, would have to show that non-T is false on general grounds, and this may be done by syllogisms in the first and second figures with a universal negative premise. But we shall seldom find an opportunity of applying this form of proof: if there be a direct proof for T we shall prefer it; if there be none, a universal refutation of non-T is usually no easier.

The only form of this method therefore which is practically important is the secondary form, which in the place of non-T, the contradictory of T, substitutes the complete

sum of all its contraries. As these contraries are all quite definite positive statements, there is more hope of being able to disprove each upon general grounds, and therefore by a progressive method. The proof that non-T is universally false which is formed by the union of these several negative proofs is then evidently a regressive argument corresponding to the positive collective proof. When T and all the contraries of T are conceived as together forming the sum of all possible relations which can subsist between S and P, the subject and predicate of T, the form of proof of which we are speaking becomes that which is known under the name of *proof by exclusion:* the truth of T then follows from the falsity of all the other members of this complete disjunction. One of the most important applications of this form is the special case of a tripartite disjunction, in which T has two contraries, i.e. in which non-T divides into two contradictories: then we get the *proof by the method of limits.* We are familiar with this proof and its very great importance in mathematics, where it belongs equally to inventive and demonstrative reasoning: every magnitude a is either equal to or greater or less than another magnitude d with which it may be compared: if it can be shown that it is neither greater nor less than d, the proposition $a = d$ is proved. In practice this train of reason generally takes another line: for the above statement presupposes that our attention has already been directed to the definite magnitude d which is proved in the end to be equal to a. As a rule this is not the case, but we only know that a is less than a second magnitude b and greater than a third c: if then we can succeed in showing that the same relation constantly holds as we diminish the value of b to β and raise the value of c to γ, the value of a must lie between two limits β and γ which are constantly approaching each other, and it will be possible to calculate this value with an approximation to the truth which may be carried as far as we please. The best known and most elementary

example is the determination of the length of the circumference of a circle by enclosing it between a larger circumscribed and a smaller inscribed polygon, and diminishing the former and increasing the latter without limit by continually adding to the number of their sides. Such forms of proof deserve our attention; they are the potent instruments by which we actually enlarge our knowledge; the development and application of this method by *Archimedes* is a greater advance in applied logic than any that ever proceeded from the merely syllogistic art of Aristotle.

212. A *sixth* method, the second *indirect progressive* method, would begin by assuming non-T, and proceed to develop its necessary consequences, and then from their falsity infer the falsity of non-T, the last step of course being regressive. I will here refer the reader back to the second direct progressive proof, and only add with reference to this indirect method that it does not matter how many true propositions may be deduced from non-T: for it is quite possible for a number of true inferences to flow even from a false proposition with respect to points whose mutual relations are not affected by the error: but a single false proposition which follows as a necessary consequence from non-T does away with its universal validity. If this consequence merely conflicts with given facts there is properly no reason for calling this proof a *deductio ad absurdum*, though the name is sometimes given to all applications of this method: all that has been done is to prove that an idea which in itself is not unthinkable nor absurd is as a matter of fact untrue. But again absurd or nonsensical is strictly speaking not that which is known to be impossible in thought, but that which conflicts with all probable suppositions, with our general feeling as to what is true, and a number of truths involved in that feeling, provable perhaps but not yet actually proved. That $2=3$ is more than absurd, it is impossible; but that the whole world is a thoughtless jest, that parents should

obey their children, that we should reward criminals and be tender to sin, are absurd assertions. I would therefore apply the name *deductio ad absurdum* only to the indirect progressive proof which develops from non-T consequences which are not impossible in thought, but which are inconsistent with a host of convictions accepted as truths and sufficiently established. This kind of proof is very constantly employed in daily life, especially whenever non-T states a thought, which is perhaps in itself correct, in too general language, i.e. when it proceeds from too wide a definition of the subject S to which a predicate P is to be attached, or from too wide a definition of this P. It is in this way that we prove the unreasonableness and foolishness of a proposed law, whether it gives or takes away rights and duties, by showing what further intolerable and monstrous consequences would follow if the proposal were carried out universally. Usually however the *deductio ad absurdum* is made to include also that form of indirect proof which deduces impossible consequences from an assumed proposition and thereby refutes it.

A particular case of this is when the development leads to a consequence which at once does away with the proposition from which we started, so that the inner contradiction which lay in the assumption of its truth of itself forces us to infer that it is false. As a simple instance we may take the indirect proof of the proposition T that on a straight line $a\ b$ in the same plane and at the same point c only one perpendicular $c\ d$ can be made to fall. Non-T then would assert that several perpendiculars were possible at the point c under the same conditions. Now assuming that this is correct, assuming further that $c\ d$ is the first perpendicular, i.e. that it makes with $a\ b$ two adjacent equal angles α, any second perpendicular $c\ e$ must, in order to be distinguished from $c\ d$, make with it at the point c some angle δ, while at the same time in order to

be perpendicular to $a\ b$ it must make with it equal adjacent angles. A look at the figure then is enough to show that the two angles $a+\delta$ and $\alpha-\delta$ must be equal, and each equal to a right angle: but if $\alpha+\delta$ be a right angle, α, being a part of this right angle, is not itself a right angle, which contradicts the original supposition that α is a right angle. The equation $\alpha+\delta=\alpha-\delta$ can only hold good when $\delta=0$, i.e. when $c\ e$ and $c\ d$ coincide. The proposition T therefore holds good: at the same point in a straight line there can be only one perpendicular in the same plane.

We are constantly led to proofs of this kind when we have to do with the simplest fundamental perceptions or propositions concerning a coherent field of thought: the impossibility of apprehending the relation of S to P otherwise than as it is expressed in T, i.e. the fruitlessness of the attempt to affirm non-T, will always betray itself by the fact that the consequences which follow from it destroy or alter the subject S or the predicate P, which were both assumed to be valid for non-T in the same sense in which they were valid for T.

213. The *indirect* proof, like the direct, admits of two *regressive* forms: these two, the *seventh* and the *eighth* in our survey, have but little to distinguish them; they bear just the same relation to the falsity of non-T that the two direct regressive proofs bear to the truth of T.

The former (the seventh) method would work back from non-T to the conditions necessary to its truth, and then reason back again from the falsity or inconceivability of these principles to that of non-T. In its application this method differs but little from the corresponding progressive method; for the principles which are necessary to the truth of non-T can only be found by taking non-T as their basis and developing them from it as its consequences, i.e. progressively. The latter (the eighth) method would start from given facts or principles and proceed to show that they cannot be founded upon non-T as their basis, but

rather expressly require the falsity of non-T. This also we shall find can only be carried out by either developing non-T progressively into its consequences, and ascertaining that if they held good they would make the given facts impossible, or by taking these given facts for basis and deducing from them, progressively as before, their necessary presuppositions: but this will very seldom be of much use, for in that case it will usually be easier to establish directly that T as such a presupposition must be true, than indirectly to establish that non-T cannot be true.

I will conclude this survey with the general remark that I believe that I have correctly distinguished in my classification the various aims of demonstrative reasoning, but that not every one of these aims has corresponding to it an equally important and equally peculiar form of proof, clearly distinguishable from all the other forms; it was enough therefore to examine in detail only those which have in practice shown themselves to be methods that are frequently applicable.

214. The reader will be surprised at the absence from my list of the proof by *analogy*; I do not believe in its existence. In all cases where we believe we can prove *by* analogy, the analogy in fact is distinctly not the ground of the conclusiveness of the proof; it is only the inventive play of thought by which we arrive at the discovery of a sufficient ground of proof: it is upon this ground, by means always of a subsumption of the individual under a universal, that we establish the necessity of the proposition to be proved. Although it will take a considerable space, I think I must consider this point in detail.

It may be regarded as a fundamental principle of analogy in the strict sense, holding good in all cases without exception, that of like[1] things under like[1] conditions like[1] assertions are true,—a statement which the mathematician further expresses in a number of special ways adapted

[1] [The German word is 'gleich' not 'ähnlich.' See note p. 327 below.]

to his various problems. It is easy to reduce this principle to the principle of subsumption: if P is true of S under a condition X, S and X may be comprehended in a general conception M, of which as such P is true; under the same M we may subsume any other S which is like the first S and subject to a like condition X; therefore the same predicate belongs to this S as to the first. This transformation, which may here seem arbitrary and superfluous, cannot be dispensed with in the case of the second principle,—of unlike things under like conditions unlike assertions are true. We may be inclined to regard this also as unconditionally true, but difficulties thicken upon us when we try to apply it. Suppose that unequal magnitudes a and b are divided by the same third magnitude c; in this case the principle will hold good; the quotients will be unequal. But take a second case: divide each of the unequal magnitudes by itself, and the principle seems to fail; the quotient in both cases is 1. Of course it will at once be urged that the condition X, to which the unequal elements a and b are subjected, is just not alike for both; for when we divide each magnitude *by itself*, we introduce the inequality again into the meaning of the condition which was to have been alike for both. But this explanation will not cover the following third case; multiply both by 0, and the product in both cases alike is 0. It cannot be denied that the operation of taking a magnitude no times has but one meaning, and does not as in the former case depend upon the value of the magnitude to which it is applied: on the other hand it may be remarked with justice that in this case the meaning of the like condition or like operation X is precisely of such a peculiar kind as to annul the inequality of the magnitudes to which it is applied. Take a fourth case; if we square the unequal magnitudes a and b, the meaning of the condition to which we subject them is again dependent upon the magnitudes themselves as in the

second case, only with the opposite result; the squares a^2 and b^2 are unequal. Fifthly and lastly the results are once more equal, for both $=1$, if we raise a and b to the 0^{th} power. In this case the condition to which we have subjected the unequal magnitudes a and b seems to be independent of their value; but in fact the raising to the 0^{th} power is a quite inconceivable operation; we must remember that in general a^{m-n} is merely another expression for $\frac{a^m}{a^n}$, and that accordingly a^{1-1}, which is equal to a^0, is identical with $\frac{a}{a}$, and therefore this fifth case is identical with the second. If we wish to avoid all these ambiguities the only way is to say that of unlike things under like conditions unlike assertions are true when the condition is of such a nature as not to affect the unlikeness of the unlike things: but that like assertions are true of them when the condition is such as to annul their unlikeness. But these two propositions are mere barren tautologies: they do not enable us to decide even so much as whether the assertions to be made will be like or unlike without a previous analysis of each case to teach us what is the general rule MP under which a and b are really to be subsumed here, and what are the definite predicates P^1 and P^2 which attach to them in virtue of the special sense in which they, as unlike kinds of M, partake of this universal P. When we have found these predicates P^1 and P^2 we see whether they are like or unlike; it is not by analogy therefore, but entirely by subsumption that the conclusion is arrived at.

215. To the third principle, that of like things under unlike conditions unlike assertions are true, a higher value may be assigned; it would in fact be inconsistent with the law of identity if an identical subject under really different conditions showed no trace of the influence of this difference, and I shall have occasion some way further on to

make use of this proposition as a not unfruitful maxim in the treatment of philosophical problems. But for the present what strikes us is the number of apparent exceptions. How could the engineer solve the problem of constructing a machine which under changing conditions regulates itself and maintains a uniform motion, if the same subject or material substratum under different conditions absolutely must exhibit different effects? A closer examination removes this objection; it teaches us that in the cases here concerned either the unlike conditions are not simple but go in pairs, or that the like subject is not simple, but a whole of various parts. But two pairs of conditions may with regard to a definite effect be equivalent, because the unlikenesses of the several members, in virtue of the definite relation which subsists between them, annul one another till the remainders are like; on the other hand various unlike conditions may so work upon the various parts of a whole that the several effects in each case modify one another till the resulting state of the whole is like. A simple body which is out of all relation to others can never receive under the impulse of a force a the same motion that it receives under the impulse of a force b unequal to a. But under the simultaneous influence of a and b it may be moved at the same speed and in the same direction as under the combined influence of c and d: if these four forces operate in the same straight line, the equality of their algebraical sum, i.e. the condition that $a \pm b = c \pm d$, is enough to give a like motion to the body; or in more general language, every motion m may be conceived as the resultant of a countless number of different pairs of components.

Now this result may be exhibited in various ways. If we regard the *sums* $a \pm b$ and $c \pm d$ as the conditions to which the body is subjected, then the conditions themselves are like one another, and the case comes under the principle that of like things under like circumstances like assertions are true: but if we leave the several forces separate, the

case seems to make an exception to the third principle. Nevertheless I should like to maintain that this third principle is universally true; for its true meaning plainly is that the sum of *all* the effects experienced by the same subject or substratum under different conditions will always be different. And so even if two pairs of conditions are equivalent in respect of one kind of effect which they produce on the same subject, it does not follow that they are also equivalent in respect of all their effects, and it is not proper to attend to the former only and neglect that part of their effect which is unlike. If a and b work upon a body in opposite directions, and c and d also in opposite directions, and if their sums or differences $a \pm b$ and $c \pm d$ are like, the body certainly experiences the like motion m, and remains at rest if $a = b$ and $c = d$; but it obviously experiences very different pressures according as it is two large or two small forces that hold it in equilibrium. Though a self-compensating machine continues to act alike under constant and under varying conditions, yet the position of its parts changes as the conditions change, and it wears out faster when it is obliged to exert its compensating powers than when it leaves them unused, the conditions remaining uniform. If full sunlight falls upon one scale of a balance suspended in a vacuum, while the other is in shadow, the equilibrium is not disturbed, but the first scale is warmed and expanded more than the other. Lastly if we multiply a first by $a\,b$ and then by $b\,a$, these conditions are certainly quite equivalent in respect of the magnitude of the resulting product, but not in respect of its structure, and $a\,a\,b$ is in any case a different combination from $a\,b\,a$. It would be easy to add to these examples, already sufficiently various, and thus to confirm the universal truth of the third principle; but after all it is but of very little use for a proof by analogy; it never enables us to establish what all analogy aims at, viz. that in a second case the same thing happens as in a first, but only brings us to the negative conclusion,

that *any* difference of the conditions in the same subject makes the likeness of the total effect impossible; what is still like in this effect, and what unlike, we can never tell without an enquiry of another kind.

The fourth principle needs but the barest mention; that of unlike things under unlike conditions unlike assertions are true is, after all that has just been said, so evidently unfounded or ambiguous, that no useful application of such a statement is conceivable.

I will only add in conclusion that the trains of thought to which the title of proofs by analogy is supposed to be appropriate do not even proceed directly from these principles, though they must be traced back to them. The presupposition on which they rest is rather that of similar things under similar circumstances similar assertions are true. Now similarity[1] is always a mixture of identity[2] in one respect and difference in another; if therefore it is difficult to base any valid inference upon the foregoing propositions which separate the mingled elements, it is still less possible to do so when the two are indiscriminately fused together in the resemblances to which appeal is made. I think therefore that I have sufficiently shown that there is no such thing as a proof *by* analogy; though I do not by this intend to deny that the observation of even remote resemblances is of great assistance to the discoverer both in detecting new truths and in finding grounds for proving given truths; for, to sum up my meaning briefly, there is no need to impugn the abstract validity of these three principles, but only their fruitfulness for demonstration. We cannot on the ground of the unanalysed similarity of two subjects transfer the predicate of one to the other, but

1 ['Aehnlichkeit.']

2 ['Gleichheit.' It is impossible to adhere to a single rendering for 'gleich.' Thus 'unlike' applied to *magnitudes* as on p. 323 might mean heterogeneous; 'ungleich' is therefore rendered there by 'unequal;' but in the rest of this passage by 'unlike.' Cp. Metaphysic, sect. 19, note.]

only on the ground of their demonstrated identity, identity at least in respect of the conditions upon which the predicate in question everywhere depends; and this always brings us back to setting down a universal proposition MP and *subsuming* both subjects under the determining conception M.

216. We have still to consider those mathematical arguments which are commonly called proofs by *strict analogy*. As the name analogy originally meant proportion, every procedure that leads back to proportion has a reasonable claim to the title; the effect of common usage however is such that when we hear of an inference by analogy we expect an argument which reasons directly from similars to similars, without needing to take a circuitous route through a higher universal. But the methods employed by mathematicians cannot be thus opposed to proof by subsumption. A proportion between four determinate magnitudes, $a:b=c:d$, is merely the expression of a fact; it only becomes a source of fresh inferences when the last two members are left indeterminate; but in this form, $a:b=m:n$, it is the expression of a universal law; it asserts that the magnitudes yielded by the problem now before us at the moment are connected together in pairs in such a way that in every pair one member is to the other as $a:b$. If we give any definite value to m and n we get a syllogism in Darii,—all the pairs of magnitudes which the problem yields (M) have the ratio P, viz. the ratio $a:b$; but m and n (the S or subject of the minor premiss) are such a pair; therefore m and n are to one another in the ratio $a:b$. No doubt this reduction to the first figure is very tedious; but we deceive ourselves if we fancy, because of the shortness of the formulated expression which the nature of the subject-matter makes possible in mathematics, that the train of thought also in a simple proportion is something shorter than that here stated. Even the simplest example of the rule of three is worked in this way. We say, if 1 pound costs two

thalers, 10 pounds cost 10×2 thalers: here we assume, what seems to us self-evident, that the ratio between the quantity of the article and the price is *always* the same; accordingly we take the ratio of the one pound to its price as a general rule and bring the ratio of the 10 pounds to its price under it as a particular case of the rule: but the dealer perhaps sells the 10 pounds for 18 thalers and thereby shows that what we assumed is not self-evidently true in all cases, but that we really had to *make* the assumption for the purposes of our calculation: further it is evident that we tacitly conceive m and n as standing for quantities of the same article and of the same unit of currency as a and b, and so in this respect also take the first case as the general rule and *subsume* the second case under it. Every general equation which exhibits one and the same content under two different forms is equally a general *rule*, which holds good only for that kind of magnitudes which, by a convention which finds no expression in the formula itself, we intend to denote by these particular letters, and for which we originally showed the equation to be valid. It is not allowable therefore to substitute for the magnitudes m and n which occur in an equation any other chance magnitudes μ and ν, and to regard the equation as still valid: we must know beforehand that μ and ν can be subsumed under the species m and n of which the equation has been proved to be true. Suppose we have proved by actual multiplication and by the argument from n to $n+1$ that

$$(1+x)^m = 1 + \frac{mx}{1} + \frac{m(m-1)}{1.2}x^2 \dots;$$

that does not give us the right to infer also that

$$(1+x)^{\frac{1}{m}} = 1 + \frac{x}{1.m} + \frac{1\left(\frac{1}{m}-1\right)}{m\,1.2}x^2 \dots;$$

for in the first formula m stood only for the class of positive

whole numbers, for which alone the proof by multiplication was feasible, and a fraction cannot be subsumed under it. If on the other hand we had found means to prove in the first instance that the binomial theorem *in the first case* holds true for the fractional exponents $\frac{m}{n}$, whatever positive value may be assigned to m and n, we might have deduced the first formula directly from this, since every whole number m may be expressed in the form of an improper fraction.

217. In conclusion I should like once more to connect what I have said with the *dictum de omni et nullo* or the law of disjunction. If S^1 and S^2 are two species of the genus M or two particular cases of the universal M, and if P may be predicated universally of M, we know that P may be predicated of S^1 and S^2 not in this universal form but in the modified forms P^1 and P^2. Now in a special case it may happen from the way in which the various predicates $P\ Q\ R$ are connected in M, that the various groups of characteristics $p^1 q^1 r^1$, $p^2 q^2 r^2$, $p^3 q^3 r^3$ which they form in the several subjects $s^1 s^2 s^3$ must be identical with one another; they then make so to say a secondary predicate Π, which may be ascribed to M itself, and which equally attaches without modification to every species of M. Thus the conception of the triangle M requires three angles $p\ q\ r$, but the various values of these angles in the various kinds of triangles always make up the same sum $\Pi = 2$ right angles; this identical characteristic Π therefore attaches to all triangles and we may at once ascribe it to any single triangle when we have simply subsumed it under its genus. But apart from such special cases the p^2 or q^2 that will be proper to an s^2 remains indefinite, with the single limitation that it must be a kind of Q, and that it must always be present, even though its value diminish to nought, in which case this nought must be capable of explanation. If this q^2 is to be determined, there must be a rule according to which the specific peculiarity of S^1 (which makes it not only a kind

of M but *this* particular kind) helps to determine the modifications of the general characteristics of M,—in this case the modification of Q; and we must assume that the peculiar nature of S^2 will follow *the same* rule in determining q^2, the modification of the general characteristic Q which is appropriate to it. If we know this rule we can determine q^2, and this is precisely the case which is called inference by *strict analogy*, though as we have seen this rests upon nothing but the subsumption of a case under the like universal rule. But when this rule is not known, we still feel inclined to find out q^2 by considering the resemblances and differences in the relation of S^1 and S^2 to each other and to M, and the procedure based upon this we usually call inference by analogy; but it only enables us to *guess* the right result, never to *prove* it. It was known by the forty-seventh proposition that for right-angled triangles the square on the hypotenuse h is equal to the sum of the squares on the sides a and b which enclose the right angle. As this relation can depend upon nothing but the general properties of the triangle, the right angle, and the length of the sides, it is a quite justifiable impulse which bids us seek an analogous proposition about the square of the subtending side for other values of the subtended angle. If we simply put the formula in the general form $h^2 = a^2 + b^2$ there is no longer any mention of the right angle; but the formula we are seeking must mention the subtended angle, for it is evident at a glance that, a and b remaining the same, h gets longer as the angle increases and shorter as it diminishes. Accordingly to make the Pythagorean formula complete we must add another term which will become nought when the included angle $\phi = 90^\circ$: and as we cannot measure h by the angle itself, but only by a length dependent upon it, or by a numerical coefficient dependent upon it that determines another length, we may set down tentatively $h^2 = a^2 + b^2 \pm m \cos \phi$. The alternative sign $\pm$ is seen at once to be needless when we reflect that when ϕ

increases beyond 90° h still increases but the cosine becomes negative; we only need the minus sign therefore in the formula. In order to find m which is as yet indeterminate we turn to the two limiting values of ϕ, $\phi = 0$ and $\phi = \pi$. In the latter case h^2 becomes equal to $(a+b)^2$ and cos $\phi = -1$; in the former case $h^2 = (a-b)^2$ and cos $\phi = +1$; both cases alike give us $h^2 = a^2 + b^2 - 2\,ab \cos \phi$. Now this formula is in fact correct for all values of ϕ, but it is as yet by no means proved; it covers with certainty only the three special values of ϕ, viz. $\phi = 0$, $\phi = \pi$, $\phi = \frac{\pi}{2}$, from which it was obtained: it would be easy to find another formula, e.g.

$$h^2 = a^2 + b^2 - 2\,ab \cos \phi \,.\, \cos^2 (\pi - \phi),$$

which would also cover them; which of the two is also satisfied by all the intermediate values of ϕ remains unsettled, till by an easy geometrical construction, with the help of the forty-seventh proposition, we decide that the formula we first took is universally true. I have dwelt upon this simple example in order to show how many subsidiary considerations are necessary before our efforts to discover new truths by the analogy of given truths can even be put into a path which promises success.

CHAPTER V.

The discovery of grounds of proof.

218. IN any demonstration of a given proposition T the most important thing is to find the major premiss G, from which by appropriate subsumption T is to follow as necessary consequence. This problem, obviously a problem for the discoverer, does not admit of any logical rule by which the solution could always be found with certainty, without counting upon the free co-operation of the sagacity of the individual enquirer. We must suppose that previous reflexion has already supplied a number of general truths, which are related to the content of the given T in such a manner as to be serviceable for the purpose in hand, and which, recalled to consciousness by the similarity of the matter in question, suggest themselves to the seeker as grounds for explaining the given proposition. But over and above this he must have the keenness of mental vision which detects among these truths the appropriate ground of proof, and sees the changes which perhaps are necessary to the subsumption of the given proposition under it, and this we must allow is to a large extent matter of native talent and not even independent of the moods of the moment. The logical relation however which subsists between the parts of a true and therefore demonstrable proposition must be able to give us at any rate such a clue as may save us from groping entirely in the dark and to some extent put us into the way of finding, after further

search of course, the ground of proof. This clue lies in nothing else than the fact which we remarked some time ago that every true universal proposition T, when we supplement and complete its subject and its predicate by all the subsidiary characteristics which are hinted at or implied though not expressed, must become an *identical* proposition. If then for the conception S, which occurs as subject in the proposition T, we substitute this completed sum of the several ideas which it contains in the forms of combination proper to them, this must include the ground which justifies the predicate; on the other hand if we substitute for P in its completeness the sum of the several ideas included in it, this must include all the requirements which the subject must satisfy in order that the proposition T may be true. I will attempt to illustrate by a few examples the use of this clue, and as discovery and proof here in fact follow the same road, I shall treat some of these examples as proofs of the given proposition T and others as instances of its discovery, i. e. of the solution of the question what relation expressible in a proposition T must subsist between S and P.

219. Suppose first that we have to prove the given proposition T, that the angle in a semicircle is a right angle. By analysis of the subject we find that by the angle in question we have to understand one whose enclosing lines start from the extremities a and b of a straight line $a\,b$ and intersect each other at a point in the circumference of a circle described about $a\,b$ as diameter. Now if the second part of this definition, which determines the position of the point of intersection e, is to be satisfied, the distance of e from c the point which bisects the straight line $a\,b$, must be equal to half this line, i. e. to $a\,c$ or $c\,b$. This requirement which follows from the definition of the subject suggests at once the one slight subsidiary construction that we need: we must draw this line $e\,c$, in order to bring before our eyes the relations upon which depends the necessity of the given proposition T. When

we have drawn $e\,c$ the triangle $a\,e\,b$ which we already had is divided into two isosceles triangles $a\,e\,c$ and $e\,c\,b$, while the angle at e is divided into two angles α and β: from the fact that both triangles are isosceles this follows, and so far this alone, viz. that the angle $e\,a\,c = \alpha$ and that the angle $e\,b\,c = \beta$; but from the way in which these two triangles make up the triangle $a\,e\,b$, $e\,c$ being common to both, and $a\,c$ and $c\,b$ falling in the same straight line, it follows further that the four angles α, α, β, β, are together equal to the sum of the angles of the triangle $a\,e\,b$. We have then $2\,(\alpha+\beta) =$ two right angles, and as $\alpha+\beta$ is the required angle in a semicircle, we have found that it is equal to a right angle.

It is not always that we can get what we want by such an easy analysis as in this very simple case: let us therefore take another case to illustrate an artifice that is very frequently applicable. We may perhaps already have got a proposition T which teaches us what is true of a subject which is *not* equal to S the subject of the given proposition, but diverges from it by a difference that can be stated; supposing then that by removing this difference we cause this subject to pass into the given subject S, and are able to show how the relation expressed by T is altered by this operation, we shall prove the given proposition T if it is true, or find the true proposition T if the given proposition is false, or if none were given at all.

Suppose the question to be what is the sum of the angles of a triangle. Assuming that the propositions concerning parallel lines and their intersection by a straight line have been established without taking triangles into consideration, we take two straight lines $a\,d$ and $b\,c$ parallel to one another, and intersected by a third straight line $a\,b$ in the points a and b. These three lines thus form no triangle, but an unclosed space; but we know S the sum of the two angles $d\,a\,b$ and $a\,b\,c$, and know that it is equal to two right angles. If we now make the line $a\,d$ turn about the point a so as to incline towards $b\,c$, there is formed between its new position

and its old one an angle ϕ, which is taken away from S the sum of the interior angles; but at the same time there is formed between bc and the line which has been deflected to meet it a new angle, the third angle which together with the remainder of S the sum of the original angles makes up the three angles of the triangle now formed, and which by the propositions about parallels is equal to the angle ϕ which was excluded from S. Thus therefore in the passage to a triangle from what is not a triangle the sum of the angles contained by the three lines loses ϕ and gains ϕ; it is therefore equal to two right angles in the triangle as before.

220. Suppose we want to prove or to find the conditions of equilibrium for a perfectly free and absolutely rigid body, operated upon at various points by various forces in various directions. In the conception of a body here employed perfect freedom needs no further analysis; as absence of every conditioning relation to others it is quite clear as it stands; only if the relations were present should we have further to determine their import: the absolute rigidity of a body means that the distance between any two points in it is unalterable.

Now if no force were acting upon this body, we should be able to say of it that it either was at rest, or was continuing an original motion at a constant speed c: we should therefore only have to set down $c = 0$ in order to express the conditions of the equilibrium intended, the equilibrium of rest. But in order to decide how the body maintains equilibrium when forces are acting upon it we must adopt the same method as in the preceding case and first see how it would move if it did move, and then negate all the conditions which would be inseparably bound up with this motion. This is not merely a useful contrivance without any logical basis; for the equilibrium we are now seeking must be conceived not as mere rest but as the negation of the movements which tend to disturb it. Now as the only

kinds of motion are motion from place to place, rotatory motion, and thirdly the combinations of these two, all we have to do in order to determine the equilibrium of the body is to consider the conditions of the two first-named kinds of motion; negate them and the possibility of the third kind is gone.

221. If we first consider only movement from place to place or movement of translation, expressly excluding all rotation, it follows from the definition of rigidity that all the parts of the rigid body must move onward in rectilinear and parallel paths and therefore with the same velocity. In whatever way a force acts therefore, if it has given to *a*, one part of the body, a velocity *c*, it must always, provided the movement be one of translation and not of rotation, have given the same velocity to *b*, any other part of the body. Hence we are able, to our great convenience, in estimating the movement of translation which finally results from all the forces acting upon a rigid body to neglect the fact that they act upon different points: we may treat them all as acting, in lines parallel to their given directions, at an arbitrary point in space, at which we suppose the mass of the body to be concentrated, and then by the known rules for the composition of forces determine the resulting movement R which they would impart to this point; the magnitude and direction of this resultant R are then identical with the magnitude and direction of the motion which the body receives under the united influence of these forces, and it remains at rest when $R = 0$. If we express this by saying that the body rests when the effects of all the impulses to motion which are brought to bear upon it annihilate one another, the proposition is an identical proposition for which no reason need be sought: our explanation however further states the condition under which that annihilation takes place, viz. the very same condition as that under which it will take place when all the forces are acting upon the same point.

222. In mechanics however it is usual not to state this condition under this form $R = 0$, but to break it up, for convenience in applying it to calculations, into three equations, which I proceed to mention, since the feasibility of a logical precept is certainly one of the questions which applied logic ought to consider. If the number n of the forces acting upon the body be considerable, it becomes laborious to find the last resultant R by first of all getting a first resultant out of two of these forces, and then a second out of this and a third force, and so on till the last force is compounded with the last preceding resultant. Moreover the angles which the direction of each force makes with that of any other, and which would have to be considered in this calculation, are seldom included among the data originally given; but where these data have to be first determined by the examination of a given state of things, it will be preferable here as elsewhere to characterise the directions of all the forces by their relations to a single common standard, instead of measuring the divergence between every two. The usual proceeding then is to lay down three axes $X\,Y\,Z$, at right angles to one another, and then to determine the direction of each force P by the three angles $\alpha\ \beta\ \gamma$ which it makes with these axes or with lines parallel to them, at the same time conceiving each force as resolved into three components parallel to these axes, which forces will according to a familiar proposition be $P \cos \alpha$, $P \cos \beta$, and $P \cos \gamma$. The three sums then made by adding together all the components of like direction, i.e. the sums $\Sigma P \cos \alpha$, $\Sigma P \cos \beta$, $\Sigma P \cos \gamma$, will be the resulting forces which tend to move the body in directions parallel to the axes X Y and Z respectively: if each of these sums as they stand be equal to nothing, the body does not move from its place in any of these three directions, and therefore does not move at all, for any movement in an intermediate direction would include a simultaneous change of place in the direction of two of these axes at least, and this has just been

denied. So instead of $R = o$ we have these three equations, $\Sigma P . \cos \alpha = o$, $\Sigma P . \cos \beta = o$, and $\Sigma P . \cos \gamma = o$, to express the condition which annihilates all movement of translation.

223. We have still to look for the other conditions which make the rotation of the body impossible. Suppose now that a straight line rotates about one of its points; then with the exception of this one point which we regard as fixed (thus making it impossible for the whole line to have any movement of translation) all the other points of the line alter their co-ordinates. The line therefore cannot rotate if two of its points have unalterable co-ordinates. But though the line be fixed along its whole length, a plane which contains it may rotate about it: then all the points in the plane which lie outside this axis alter their co-ordinates: the rotation of the plane therefore becomes impossible if any point in it outside the axis be fixed, or in general if the three angular points of a triangle drawn anywhere in the plane be fixed. The same condition is obviously sufficient to make rotation impossible for a rigid body, every point of which is at an unalterable distance from every point in a fixed plane taken at will in it. The condition which prevents rotation therefore might be expressed by saying that the three angular points of a triangle drawn anywhere within the body do not alter their co-ordinates. But the proof that this condition was fulfilled would not be at all a convenient one: in order to prove it by applying the previous three equations to each of these three points we must be able to prove what is the resultant effect at each of them of all the forces acting not at this point but at other points: but this, as will easily be seen, is the very thing that we are still trying to ascertain. We must take another course therefore, and, since the position of the triangle just mentioned is perfectly arbitrary, the course which most naturally suggests itself is to dispose its three angular points in the three axes $X Y Z$, by reference to which we have already determined the directions

of all the forces in operation: but the position in each axis of the angular point which we place in it is also perfectly arbitrary: we may therefore regard every point in each axis as a point whose position is unalterable, i.e. we may regard the three axes themselves as three fixed lines, in relation to which, if rotation is to be excluded, no point of the body can change its position and distance. If finally we consider the three axes as three dimensions which lie within the body itself, or as identical in position with three series of points in the body at right angles to one another, it follows from the definition of rigidity that the fixity in space of these series of points is all that is required to make any change of place impossible to the remaining points of the body. The problem therefore reduces itself to showing that all the forces in operation are unable to impart a rotatory movement in any direction to any of these three series of points, or to any of the three axes $X\,Y\,Z$ now conceived as capable of moving out of their previous direction.

224. This last way of treating the matter however would not serve as a convenient basis for calculation except when the directions of all the forces concerned passed through the three axes. This will not generally be the case: in order to take into account those forces which when produced go past those series of points without cutting them, we must substitute for the three lines three planes intersecting each other at right angles, each of which will therefore include two of these axes: the direction of each force produced if necessary must cut one of these planes. The problem now is to show that all the forces in conjunction are unable to cause either the planes $X\,Y$ and $X\,Z$ to rotate about X, or the planes YZ and YX to rotate about Y, or the plane $Z\,Y$ and $Z\,X$ to rotate about Z. Let us consider the conditions of rotation about Z. Any force P acting in any direction upon a point of the body whose co-ordinates are $x\,y\,z$, and making with the three axes the angles $\alpha\ \beta\ \gamma$, can as before be decomposed into three forces $P\cos\alpha$, $P\cos\beta$,

$P \cos \gamma$, parallel to the three axes. The last of the three we need not consider here; it could only cause a movement of translation in the direction of the axis Z, which is already excluded by the equations of § 222, or a rotation of the plane $X\,Y$ about X or Y, which also need not be considered at present. Of the two other forces $P \cos \alpha$ is perpendicular to the plane $Z\,Y$ and $P \cos \beta$ to the plane $Z\,X$; the two tend, as is shown by an easy construction, to cause the planes $Z\,X$ and $Z\,Y$, and so the body in which these two planes are immoveably united, to rotate in opposite directions: the direction of the rotation which actually results would therefore depend upon the difference between the two forces. Not simply upon their difference however, for a proposition which at present we will only allude to, teaches us that the rotatory effect of a force which is perpendicular to a line is to be measured by the product of its intensity into the distance of its point of application from the axis of rotation. For the force $P \cos \alpha$ this distance is y, and for the force $P \cos \beta$ it is x: the difference of the products $P \cos \alpha$ and $x\,P \cos \beta$, or the difference between the two momenta, must be equal to nought if P is to cause no rotation about the axis Z. We must repeat the same considerations with regard to all the forces concerned, and we finally get, as the condition which prevents all rotation about the axis Z, the equation

$$\Sigma\,(y\,P \cos \alpha - x\,P \cos \beta) = 0.$$

The other equations which make rotation about the axes X and Y impossible, will obviously, as the three directions are perfectly homogeneous, be of the same form; and, since even artificial aids to memory are not beyond the province of applied logic, I will remark that the equation for non-rotation about an axis never contains the elements which refer to this axis, but consists of the sum of the differences of two products, each of which unites a component force in the direction of the second axis with that co-ordinate of its point of application which is parallel to the third axis. The

formula $\Sigma\,(z\,P\cos\beta - y\,P\cos\gamma) = 0$ annihilates rotation about X; the third formula $\Sigma\,(x\,P\cos\gamma - z\,P\cos\alpha) = 0$ annihilates rotation about the axis Y.

225. The proposition about the equilibrium of rotatory forces which we made use of in the preceding discussion is easily arrived at in the domain of statics by a slight device which reduces the question to the composition of motions. If I now select another mode of proof, I do so of course with no idea of improving the science of statics; I only adopt a treatment which is as far as possible independent of all merely happy contrivances, in order to illustrate the way in which the grounds of proof are brought to light by the analysis of the problem itself. If the rigid line $a\,b$, whose length we will call n, rotates about its fixed extremity a, this implies that all its points describes a segment of a circle $\rho\,\omega$ with the same angle ω and with a radius ρ, which for each point is equal to its distance from the point a. If now a force W acts upon the point b, and causes b, in whatever way, in the time t to pass through the segment $n\,\omega$, it must likewise have compelled any other point in the line at the distance ρ to describe in the same time t the segment $\rho\,\omega$: and conversely, any force which applied at the point ρ has caused this point to move through the small segment $\rho\,\omega$, has necessarily compelled all the other points in the line to describe segments corresponding to their distance from a. We now ask what must be the nature of the two forces P and Q in order that when they are brought to bear at the points p and q respectively they may produce precisely equal results, and accordingly when acting in opposite directions upon the line $a\,b$ may prevent each other from making it rotate. Now the conception of rigidity, i.e. the conception of the simple immobility of a, is too far removed from conceptions of movements to tell us how the latter would be affected by the former: we should have first to conceive rigidity itself as the result of movements, in order to make it homogeneous with the

other movements upon which it is to exercise a restraining influence. Further it is impossible to compare P and Q so long as they act under different circumstances whose modifying power is yet unknown: we can only estimate them by velocities ϕ and ψ which they would impart under perfectly similar conditions to a perfectly similar moveable object: and lastly though P and Q may be applied at the single points p and q, they cannot operate upon them alone; in order to set up or to hinder a rotation, the effect of each must spend itself over all the points in the line $a\ b$, and we must know the mode of this distribution before we can understand how the effect of the one can annihilate the simultaneous effect of the other at every point in the line.

226. These requirements we may satisfy in the following way. Suppose that $a\ b$, which is equal to n, is first of all a perfectly free rigid line, consisting of an infinite number n of homogeneous points which are compelled (how does not concern us) to maintain unchangeable distances from each other. Suppose that a number n of equal and parallel forces operate perpendicularly upon this line so as to give to each element of it the velocity ω; then the total force W, equal to $n\ \omega$, will urge the whole line forward, all the points moving in parallel directions. This movement of translation passes into a rotatory movement when we give to the various points of the line various counter-velocities, which must be conceived as at right angles to $a\ b$ not only at the beginning of the rotation but at every subsequent moment. To the extremity a we assign a counter-velocity $-\omega$ by which it becomes the fixed point which our problem requires; to the point b we give a counter-velocity which $= 0$, so that it maintains undiminished the velocity ω imparted to it by W; the intermediate points must meet so much resistance as will leave to each point ρ, whose distance from the fixed point is ρ, a residual velocity whose amount is already known, viz. the arc $\frac{\rho}{n}.\omega$,

whose length is to ω, the path of the free end, as ρ is to n: the sum of the velocities of all the points ρ, from $\rho = 0$ to $\rho = n$, must be equal to $\frac{n \cdot \omega}{2}$. Now a force P, which would give to a free element the velocity ϕ, would give to an element p in our rigid line the velocity $\frac{p}{n} \cdot \phi$, if p were subject to the above-mentioned resistance but able to move by itself; but as it cannot move by itself, the impulse imparted to it must distribute itself over the whole line. However this distribution may be effected, we already know the result; it can produce nothing but a rotation of the whole line, in which every point ρ receives a velocity proportionate to its distance from the fixed point and the sum of all the velocities is $\frac{p\,\phi}{2\,n}$. Every point ρ therefore receives the velocity $\frac{\rho}{n} \cdot \left[\frac{p}{n} \cdot \frac{\phi}{n}\right]$. Precisely similar statements may be made about a second force Q, which would give to a free element the velocity ψ, but to an element q of the line which is fixed at one end would give the velocity $\frac{q}{n} \cdot \psi$; when applied at q it would give to any other element ρ of the line the velocity $\frac{q}{n} \cdot \left[\frac{q}{n} \cdot \frac{\psi}{n}\right]$. Now if these two forces operating at p and q or the two velocities produced by them are to be such that when acting in the same direction either would annihilate one and the same third movement of the line, or that when acting in opposite directions they would counterbalance each other, then for any point ρ the two expressions which we have just found for their effects must be equal to one another, and so therefore $p\,\phi = q\,\psi$, and $\phi : \psi = q : p$. In other words the length of leverage must vary inversely as the strength of the force.

227. The following would be a very plausible, and yet an inadmissible way of deducing the same proposition.

Suppose that at the same point m of a lever playing in a vertical plane two equal forces P and Q are acting in opposite directions; it is self-evident that under these conditions equilibrium will be the result. Now if, as is commonly done, we imagine Q to be a weight, suspended by a hook or cord at m, and P as a strain exerted from above, we tacitly assume that it is indifferent whether of the infinite number of infinitely thin perpendicular strips into which Q may be decomposed in thought each severally grapples the point of the lever which it would touch if produced, or whether all these several forces operate upon the lever only through a single representative which unites them all, viz. the cord. Once assume this, and it must also be indifferent whether we conceive Q as *one* body, or as divided perpendicularly by a geometrical plane into two halves which touch one another at the surface of section, and each of which is attached to the lever by a separate cord which unites all its forces in one resultant. If then m was the distance from the fulcrum of the original point of attachment, $m - x$ and $m + x$ are the corresponding distances of the new points of attachment of these two cords. In other words equilibrium is preserved when two forces each of which is equal to $\frac{Q}{2}$, and whose sum $= P$, are applied at equal distances right and left from the attachment of the opposite force P: for the cords themselves, or their tensions, are now the forces which are directly applied. Now so long as these tensions are the resultants of the forces of gravity united in the two bodies $\frac{Q}{2}$, it is evident that it is quite indifferent how these bodies $\frac{Q}{2}$ are shaped in other respects, indifferent therefore whether they still touch one another as before, or whether by increase of their length and diminution of their thickness they become two separate bodies with a space between them. If we follow out this

line of thought we see that it is quite possible to carry the displacement of one $\frac{Q}{2}$ to the left and of the other to the right by equal distances x as far as we please, till at last x becomes equal to m: when that is done one $\frac{Q}{2}$, say the one that was displaced to the left, has reached the fulcrum a, and no longer produces any effect upon the lever: the other $\frac{Q}{2}$ has arrived at the distance $2m$ from the fulcrum, and the equilibrium is still preserved under the condition that P, which $= Q$, operates at the distance m from the fulcrum, while $-\frac{Q}{2}$ operates at the distance $2m$.

But though this exposition brings the matter before us very plainly, it is nevertheless absolutely inconclusive. So long as x was less than m, the $\frac{Q}{2}$ that was moved away to the left had still a recognisable and intelligible influence upon the equilibrium of the lever; we could still see plainly that it together with the other half that was moving away in the opposite direction made up the force that was sufficient to counteract P: but so soon as x becomes equal to m, and the effect of this $\frac{Q}{2}$ altogether ceases, there is a break in the thought: for one of the points of relation has vanished, and our whole reasoning was founded upon its relation to the other. For when we first applied Q at the point m itself, and then disposed the two halves of Q symmetrically on either side of m, what we inferred held good in the first instance for the free line $a\ b$, which was supported at m by the force P: the fixing of the end a was not contemplated at all: though of course the same inferences held good also for the case when a was fixed, so long as it could be proved that, irrespective of this, equilibrium was maintained by the way in which the weights

were distributed; for if equilibrium was maintained thus it could not be disturbed by the fact that a was over and above this regarded as fixed. But so soon as the influence of one half of Q vanishes, we no longer have equilibrium on the same grounds as before, and it is by no means self-evident that the vanished condition is exactly replaced by the fixing of the end a. We should in fact need for this special case to find a subsidiary proof which should show that a being fixed the effect of the half of Q was all along getting less and less as it approached a, and that equilibrium was nevertheless maintained; therefore it would continue to be preserved when the influence of this weight was reduced to nothing, while the other was removed to a corresponding distance. But if we examine it, we see that this subsidiary proof would in reality be the proof of the main question, i. e. it would be the proof of nothing less than the proposition that the power of equal forces to move a lever varies as their length of leverage. This mode of statement therefore, however plainly it brought the proposition in question before us, did not in the least prove it, but only assumed it in a circle which it is easier to recognise than to state briefly.

228. Complicated mechanical problems cannot always be solved by directly compounding all the forces in operation so as to arrive at their final resultant; we often have to state certain universal conditions which it must satisfy, or certain limits within which it must keep: with these assumptions then the several data of the given case supply means for the complete determination of the result. These methods, among which we need only mention the application of d'Alembert's principle, are quite invaluable and cannot be dispensed with: but as they do not clearly show the history of the result which we calculate by them, we still feel a wish to employ direct constructions so far as possible. I will mention in connexion with the preceding problem of the equilibrium of rotatory forces that of the motion which they

generate when they are not counteracted. The rule for calculating it is reduced to these two very simple propositions: (1) if a force acts upon a body that is able to move freely, its centre of gravity takes the same rectilineal motion which the whole mass of the body would take if it were concentrated at the centre of gravity and there acted upon by the force: (2) at the same time the body takes the same rotatory motion which it would receive from the same force if its centre of gravity were fixed. Now in this very neat division of the result there lies a paradox. For if the direction of the force passes through the centre of gravity, there arises according to the second proposition no rotation, but only a rectilineal movement of translation, and yet we should suppose that in this case the force was acting upon the body under the most favourable conditions: but if the direction does not pass through the centre of gravity, in which case the force would seem to act under less favourable conditions, there follows not only the entire previous result but also a rotation, which strikes us as an addition without any obvious reason. If the compound velocities of the various parts of a body which is at once moving onwards and rotating be decomposed into velocities in the direction of its rectilineal course and velocities in the directions perpendicular to this and to the axis of rotation, the sum of all the former components, each multiplied into its differential-mass, is equal to the product of the whole mass multiplied into its rectilineal velocity; and we easily convince ourselves that when the body is at once rotating and advancing, though the several elements have various velocities in the direction of its course, yet the sum of all these velocities is neither increased nor diminished, but only otherwise distributed than it would be in the same mass advancing without rotating. But the other components remain, and though they have opposite signs for the two halves of the rotating body, yet they do not on that account annihilate each other: they are motions which

actually occur, and we are forced to ask where they come from.

229. It is sufficient to answer this question in the simplest conceivable case. Let a and b be two equal masses, which we conceive to be concentrated at their centres of gravity: suppose that they act upon each other so as to remain always at the same distance $a\,b$ from one another: we may say then that a and b are united by a rigid unchangeable line $a\,b$ which has no mass. In order to simplify the figure to be drawn, conceive $a\,b$ to be so fitted into the angle of two rectilineal axes which intersect at O that a lies upon the axis X and b upon the axis Y: at starting then we have, for the mass a, $x = O\,a$ and $y = 0$, and for b, $x = 0$ and $y = O\,b$, while for the centre of gravity of the system $a + b$, which lies in the centre of the line $a\,b$, we have $x = \frac{O\,a}{2}$ and $y = \frac{Ob}{2}$. We will now suppose that a certain velocity is imparted to the mass a in the direction of the axis X, and that $a\,\alpha$ is the path which it would traverse in an indivisible moment of time under this impulse if it were free. As no force is acting directly upon the mass b, it would then remain at rest, and the line $\alpha\,b$ which expresses its distance from a which has moved away would be longer than the original line $a\,b$. But the forces in operation between a and b, which according to our assumption maintain the distance $a\,b$ unaltered, oppose themselves at every moment to the beginning of this elongation the measure of which would be $\alpha\,b - a\,b$, and prevent it, by making the two bodies approach one another in the direction of the line at the extremities of which they would be found if the elongation actually took place. Since neither of the two masses can one-sidedly compel the other to follow it, but both masses, being assumed to be equal, must by the principle of the equality of action and reaction displace each other to the same extent, we shall find their new positions a^1 and β by cutting off from the line $\alpha\,b$ the length $\alpha\,a^1$ equal

to $\frac{\alpha b - ab}{2}$, and from the line $b\,\alpha$ the length $b\,\beta$ also equal to $\frac{\alpha b - ab}{2}$. If from α^1 we let fall an ordinate, which we will call dy, upon the axis X, and from β let fall a perpendicular, which we will call dx, upon the axis Y, we have two equal and similar triangles, and thus we get for α^1 and β, the two extremities of the now displaced line $a\,b$, the ordinates dy and $O b - dy$ respectively; and therefore for the centre of gravity, which is still the centre of this line, we have $y = \frac{O b}{2}$: but this was also the ordinate of the centre of gravity before any velocity was imparted to it: the centre of gravity therefore has received an impulse to move in a direction parallel to the axis of X, i.e. in the same direction in which a would have been impelled to move if the force had been brought to bear directly upon it. At the same time we have for the extremities α^1 and β the abscissae $O a + a\alpha - d x$ and $d x$ respectively, and thus for the new position of the centre of gravity we have the abscissa $\frac{O a + a \alpha}{2}$; therefore, since the abscissa of its original position was $\frac{Oa}{2}$, it has received half of the velocity $a\,\alpha$ which the force applied to a tended to impart to a, and this is precisely the velocity which the same force would have imparted to the whole mass of the system (which is $a + b$ or $2\,a$) if that mass had been concentrated at the centre of gravity and the force applied to it there.

These considerations apply to the first instant of the whole motion, in which (as is usually assumed) the force applied to a, working instantaneously, gave it a certain velocity without any lapse of time, and in which the corrective reaction of the forces at work between a and b also took place without lapse of time. Since from this instant no external force any longer operates, all the

motions produced will simply continue according to the law of persistence, only the internal forces that act between *a* and *b* have to be continually at work in order to prevent *a* and *b* from flying off at a tangent, and to maintain them at a constant distance from their centre of gravity; they thus generate a rotation which is circular in relation to this point, and since they are continually diverting the two masses from their momentary direction into another without any breach of continuity, the rotation takes place uniformly in a circle and with the same constant velocity with which both masses are impelled in a straight line at the first moment.

Lastly if we move back $\alpha^1 \beta$, keeping it parallel with itself, till its centre of gravity coincides with that of *a b*, the two lines will make with one another at the centre of gravity an angle ϕ equal to that which *a b* would make with *a b* at the point *b* if *b* were a fixed centre of rotation and the external force had only had to move the mass *a* under the condition that it should always be at the same distance *a b* from *b*. The length of the curve which *a* would then have described would have been $ab . \phi$; the length of the curve actually described by *a* in rotating about the centre of gravity which we regard as fixed is $\frac{ab . \phi}{2}$; and this is precisely the velocity which the force must impart when it has at the same time to move the mass *b* in the contrary direction. From this we see that a momentary external force, whether its direction pass through the centre of gravity or not, always produces in the body the same sum of movements of translation: the rotation which is added in the second case is due to the internal forces which act between the parts of the system moved. But these forces are by no means inoperative even in the first case where no rotation occurs: but in the first case their only effect is to cause the several parts of the mass, which are arranged in a straight line at right angles to the direction of the motion

imparted, to maintain this order during the onward movement, an effect which reveals itself in no relative movement of the parts about their advancing centre of gravity so long as we proceed upon the assumption that the body is absolutely rigid; but it would at once announce itself in such movements if we conceived say three equal masses *a b c* united to one another by *pliable* cords and then imagined an impulse to be brought to bear upon the centre of gravity of the whole system which lies in *b*.

230. In the analysis which is required for the discovery of the grounds of proof we try not only to bring out the elements which are essential to the truth of the consequence to be proved, but also to eliminate those that are unessential for that purpose. For instance it is not uncommon in answering statical and mechanical questions to start from the supposition of a rigid line without mass. Now it may be granted that in the conception of a finite straight line the characteristic of finiteness implies the constant contact of each point with two neighbouring points, and the straightness implies that the line is rigid and cannot bend : only as a mere geometrical line it is not an object that could be set in motion by forces at all; the capacity of being affected by forces belongs to the lineally arranged *mass* only, and it is only the forces exerted upon one another by the minute components of the mass that actually give to this material line the rigidity and unalterable length which is merely demanded in the geometrical conception.

A *line without mass* therefore is not a happy expression, and does not in fact convey that which we really mean and upon which we build in carrying out such enquiries. A line must undoubtedly have mass if forces are to cause it to rotate about its extremity, but with a view to the laws which regulate the effect of these forces it is only necessary that the mass be the same at any cross-section of this material line; any irregularity in its distribution would constitute a special case, in determining which we should have to

apply with reference to these special data the laws of that simplest case when we have the problem in its purest form; on the other hand it is perfectly indifferent for these laws how great this mass is; the proportions between the forces and the leverages necessary for equilibrium are precisely the same whether the lever be thick or thin, whether its specific gravity be greater or less. When we speak of a line without mass therefore we do not strictly speaking set down its mass as nothing but rather as a unit, and further as a unit to which any value great or small may be given, and which disappears from our further calculations just because as an equal factor of all the terms that stand in proportion to one another it does not in the least contribute to determine or to alter the relation which subsists between them. This was the thought upon which the foregoing exposition rested. The line $a\,b$ was conceived as a line of mass, and every one of its points as a differential of the mass: it was only this that made it possible to speak at all of a force W acting upon $a\,b$, and to set down this force W as equal to $n\,\omega$, equal to a sum of individual forces each of which was such as to give the velocity ω to the differential of the mass. But we should have gained nothing by constantly taking count of the mass in our calculation; only the value of ω would have come out differently according as the mass of the line or of every one of the n parts of it which we distinguished was conceived as greater or smaller; the relations between P and Q would have undergone no change so long as both were always related to the same mass. The division of the labour of proof therefore which is here introduced does not consist in first putting mass altogether out of sight and proving the law in question for the line without mass, and then enquiring in the second place what becomes of this law when mass is given to the line; on the contrary we took count of this mass at the first step, but found that its magnitude has no influence upon the general form of the law: upon this ground then we may proceed in a second

enquiry to ask how differences in the magnitude and distribution of the mass affect the absolute values of the magnitudes which are to be determined by the law. As soon as we take this line without mass literally and think of its being moved, we become involved in absurdities through which we can never fairly make our way, since the combination of ideas upon which they rest is in itself an impossible one. What is supposed to happen when one extremity b of such a line receives a velocity c? It cannot separate itself from the rest of the line, for then it would not be the line, but only the free point b that was moved: but as the line has received no motion how can it follow the point? It may perhaps be supposed that this line would rotate: then the point b would have to communicate its velocity to the other points, and that in degrees, more to the nearer and less to the remoter points; but we cannot see how this is to be measured, for all the forces are absent here which operating between the minute parts of a mass might cause the impulse received by one part to extend itself to the rest of the series, so that every member of it might at every moment receive a definite proportion of the impulse. Finally as there is here no reason for such an apportionment of the effect we might instead of this come to regard the whole line $a\,b$ as a unity so closely bound together that every part of it, separable only to our thought or sense, immediately assumes the same states that are set up in any other part: setting aside the question whether every part of the line would then receive the whole velocity c or only $\frac{c}{n}$, the result would at all events be that the line $a\,b$ remains at rest when b receives the velocity c and the other extremity a receives an equal velocity $-c$. All these absurdities are avoided by the admission that only a line that has mass can be moved, not a line that has no mass.

231. In the subsidiary processes also, the substitutions and transformations by which we endeavour to make the

given circumstances accessible to our judgment, we have to avoid suppositions to which, however much they may help the imagination, no real meaning can be given. To illustrate this I will mention a proof which is often employed to demonstrate the *parallelogram of forces.* The body is supposed to move in a plane from *a* to *c*, and at the same time this plane is supposed to move from *a* to *b*; and in this way it is fancied that the course of the body from *a* to the end of the diagonal of the parallelogram *a b c d* has been ascertained. This involves two assumptions which are not expressed but to which expression must be given; they are first the assumption that the motion of the plane will not interfere with the motion of the point in the line *a c*, and secondly that the moving plane will carry with it the whole line *a c* together with the body. Now an empty surface in motion is sufficiently far removed from anything that can actually occur, but it is still harder to understand how a body can stick to it while it moves. And yet it is very necessary that it should so stick: for if the body be upon a very smooth table and we give it a push towards *a c*, giving the table at the same time a push towards *a b*, the body will not go with the table but will part company while the table flies away from under it. But if we supply this necessary condition, i.e. if we say that the body continues to move undisturbed towards *c*, while *a c* at the same time is compelled to move towards *b* and to take the body with it, the whole proposition becomes an empty tautology, and that which is assumed is precisely that which was to be proved. It must rank then only as one of the means which may be employed to give us a picture of an already demonstrated truth.

232. Among the numerous other proofs of the same proposition several proceed from a common starting-point which is of interest for the logician. They begin with a statement of the special case in which two equal forces *a* and *b* impel the body in two directions, and it is regarded

as self-evident that the direction of the resulting motion will bisect the angle between these two directions. But this assumption includes the further assumption that if the forces be unequal the resultant will divide the angle into two unequal parts, and since it is impossible that the kind of this inequality should be independent of the relation between the magnitudes of the forces, seeing that the fact of the inequality depends upon it, this assumption rests on a more general assumption, viz. that if two conditions *a* and *b* tend to give each a different form to a result *c*, the recognisable influence of the two in the actual form of the result will be proportional to their magnitudes; if then *a* and *b* are equal, *c* will be as far removed from the result which would have followed from *a* alone as from that which *b* alone would produce. Now I cannot see why we should appeal to this proposition once only when we are introducing the proof, and then conduct the proof itself by other complicated considerations: whatever be the forces *a* and *b* and the degree of their inequality, we may say universally that the extent to which the moved point is deflected by the force *a* from the path of the force *b*, and by *b* from the path of *a*, must vary directly as the diverting forces. In order to turn this logical proposition to mathematical use we should need first to determine how the two deflections are to be measured. The nature of the question does not invite us to apply the ordinary method and to let fall perpendiculars from the direction of the several paths upon the resultant or from the latter upon the former: all three paths are considered not as empty directions in space, but only as loci which would include the successive situations of the moved point.

The following treatment is the only one suggested by this last remark. Let α and β be the two points in the paths of *a* and *b* respectively which the moved body would have reached in the same time *t* if it had followed the force *a* only, or *b* only; and let ρ be the point in the resultant at

which the body arrives in the same time t under the combined influence of a and b: then $\rho\,\alpha$ represents the deflection from the path a effected by the force b, and $\rho\,\beta$ the deflection from the path b by the force a, and $\rho\,\alpha : \rho\,\beta = b : a$. Since we can only estimate the magnitude of the forces a and b by the space which they cause to be traversed in the unit of time, the ratio $a : b$ is also for the unit of time the ratio of the spaces traversed in the direction of a and b respectively; but it must also have this meaning for any time t and for any part of t; for since a and b are regarded as forces that operate for a moment only, the movement in the direction of the resultant must take place with constant velocity and in a straight line: the length which is traversed in the direction of the resultant therefore will always be proportional to the space traversed in the directions of a and b within an equal time t, and the lines $\rho\,\alpha$ and $\rho\,\beta$ which represent the deflections will form the third sides of triangles whose two other sides increase in the same constant ratio.

233. But this proportion tells us nothing about the absolute magnitude of $\rho\,\alpha$ and $\rho\,\beta$; they satisfy the proportion so long as they are $m\,b$ and $m\,a$; the value of this m would still have to be ascertained. Now there is nothing in all the data of the problem that can help us to determine this: none of them could have any influence upon it except the magnitude of a and b, including the ratio of a to b, and the size of the included angle; but the suppositions already made seem to have taken full count of the influence of these elements; and it is quite impossible that anything outside the data of the problem can contain the grounds of something that is to flow directly from the problem itself. In cases of this kind the logical course must always be to search for the *most probable supposition* that satisfies the requirements. The meaning to be attached to this expression would be very hard to define in general language; and my sole purpose in treating this problem is to make up by an illustration for the want of a precise determination of

the general conception. The most probable supposition will set down that which in virtue of its nature or magnitude is the minimum that makes possible the relation which we know must subsist, and which, if it were to subsist under other conditions or with other subsidiary characteristics than those we take, would necessarily furnish special reasons for inferring them, which reasons are here absent. In the case before us the proportion $\rho\alpha : \rho\beta = b : a$ must always subsist; therefore m cannot be nought; but in order that it may subsist it is enough to set down m as equal to 1; and this value of m may be regarded as by its nature the minimum that satisfies the requirements; for any greater or smaller value, as $m = 2$ or $m = \frac{1}{2}$, may be treated as $m \,.\, 1$, i.e. as so many repetitions of the unit with the vanishing of which m itself vanishes and with it the whole relation. Unity is the only value of m which affirms that the required relation actually subsists in such a way as to enable the other special values of m to be effectively introduced as further specific characteristics, in case there be any reason in the nature of the content under investigation for preferring one of these values rather than another. Where as here there is no such reason we fall back upon the supposition that $m = 1$, a supposition which in any case is necessary, and therefore is the most probable supposition; for under all circumstances, even if m had some other value, it would hold good at the same time with that value and equally satisfy the required proportion. Let us then make the assumption and construct the figure accordingly; i.e. let us from α, the extremity of the path traversed in the time t in the direction of a, describe a circle with radius equal to the path traversed in the same time towards b, and from β describe a circle with radius equal to the distance traversed towards a; then these circles will cut one another in the diagonal of the parallelogram formed by a and b, and the direction and length of the resultant are both determined at once.

234. But even when analysis has failed to detect any grounds in the data of a problem for any other than this most probable supposition, it is seldom possible to be absolutely certain that such grounds are not there, and might not be revealed by a more careful analysis. And so no pains must be spared either to confirm the supposition adopted by subsidiary proofs upon a different line, or to establish it indirectly, i. e. to exclude all other suppositions by showing the contradictions in which they involve us. We will take this further step then.

It seems self-evident that the resultant of two forces can never be greater than their sum; it attains this maximum when they both act upon the body in the same direction, and when the included angle therefore is nothing. It has been objected to this proposition also that it is after all not self-evident that when a second motion *b* is joined to a motion *a* in the same direction *b* is simply added to *a*; for it is conceivable that the nature of motion or that of the bodies subject to it involves conditions which might even in this case make the resultant greater or less than the sum of the two. This objection seems to me unfounded, especially as applied to the case before us. In the first place when two motions in the same direction are given at the same time to one body, we may continue to regard them as two separate motions, but it is only because we choose so to regard them. They were two motions outside the body: they may have been imparted to it for instance by two other different bodies. It may be also that in the physical act of transmission from one body to another the motions may lose or gain something: but we are here speaking not of the mode of transmission, but of the velocities, so far as they already *have* been transmitted to the body in question. In this body, here considered simply as something moveable, without regard to all its other peculiar properties, the two do not need to be combined into one, but they are absolutely *one* from the be-

ginning, and the resulting velocity is the sum of the two as surely as any velocity is what it is. But suppose the body already has a motion a when the second b supervenes; this could not make any difference unless the body violated the law of persistence and altered its motion every instant: for if it does not alter its motion, i. e. if at the time t it is in precisely the same condition as at the time t^0, the motion b which supervenes later must combine with the still subsisting motion a just as it would have done at the time t^0 if both had begun together. We may regard it as established then that the resultant R of the two forces a and b acting in the same direction can only be $a+b$. Of course this does not directly help us to estimate the result of forces whose directions diverge and make an angle ϕ. Meantime however it is at all events evident that the resultant cannot increase with the divergence; for then it would be least when the directions are the same, whereas we have just seen that it is greatest then, and greatest when they are opposite, whereas it is evident that it is least then. But it is equally impossible that it can be independent of the magnitude of the angle ϕ; and so it must necessarily diminish as ϕ increases, and we may now say that for forces of any direction the resultant R is either equal to or less than $a+b$.

This conclusion again which is still indefinite may be brought within narrower limits, if we apply the important general principle, that objective conditions are independent of variations in our cognitive procedure. When various momentary forces to any number we please are brought to bear at the same time upon a moveable point, the total result which actually arises can only be one, and therefore cannot alter with the various arbitrarily chosen series in which we in our minds first arrange the simultaneous conditions by pairs, and then again combine the several results thus obtained. It must be the same in the end therefore whether we first get the resultant R out of a and b

and then try to get a second resultant out of R and $-a$, or whether we combine a, b, and $-a$ so that, a and $-a$ obviously cancelling each other, b is left as this second resultant. The conception of R therefore as the resultant of a and b implies that if we again take as components R and a with its original direction reversed and calculate their resultant by the same law by which we get R from a and b, we must come back to b; and so R and $-b$ combined will bring us back to a. And this consideration holds good universally, and quite independently of the still unknown law which regulates the dependence of the magnitude and direction of the resultant upon the magnitude of the component forces and the included angle. From this then it follows that each of the three forces or motions a, b, R is under the circumstances stated above the resultant of the other two, that each is therefore less than or at most equal to the sum of the other two; whence it follows that the three may be combined in a triangle, which contracts itself into a straight line only in the limiting case where one is equal to the other two.

But as thus obtained this familiar proposition only expresses a relation between the lengths of a, b, and R; we must also make out the relations between the angles for which this relation holds between the sides. If a and b and the included angle ϕ be given, the length of R, as yet unknown, is completely determined; for these given elements therefore there is only *one* possible triangle to be made out of a b and R. Conversely, given a triangle with a b and R for sides, there is but *one* angle ϕ which the forces a and b can make so that R shall be the length of their resultant. Geometrically R in the triangle increases, if a and b are constant, as the opposite angle ρ increases; mechanically, as the resultant of a and b, R diminishes as the angle ϕ increases; between the angle ρ in the triangle therefore and ϕ the angle at which the forces diverge from one another there must subsist some definite relation which we want to ascer-

tain. In the triangle made up of a b and R, R has not the position which it must assume when it represents the resultant; in the latter case all three lines must start from a common vertex A, and it may be taken as self-evident that R must lie in the angle between a and b. Let us suppose then that a and b are two forces, as yet indefinite in magnitude, put together so as to make any angle ϕ; and that R the resultant, also as yet arbitrary in length, divides this angle into any two parts, C being its other extremity. Now as the mechanical relations of which we are here in search must be independent of the absolute position of the lines in space, we may first shift the whole system of the three lines a b and R so that the vertex A falls upon C, and then turn it, in the plane in which it lies, about C so that the forces a and b, which in their new position may be denoted by a^1 and b^1, proceed from C in directions parallel but opposite to their former directions. Then evidently the resultant R^1 of these forces a^1 and b^1 must be both in position and magnitude identical with R, only opposite in direction. Thus then the direction of the resultant is determined; it must be the diagonal of a parallelogram formed by the intersection of the forces a and b^1 on the one side and the forces b and a^1 on the other, or by their meeting in a common extremity, or by their being produced to such an extremity. But if the lengths of a and b are given, the length of R is also determined, it must be the third side of a triangle whose other sides are a and b^1, which $= b$, or b and a^1, which $= a$; it is therefore the diagonal of the parallelogram formed by the lengths of the forces themselves. The figure then shows that the angle ρ subtended by R in either of these triangles is the supplement of the angle which the forces make with each other, i.e. that $\phi = \pi - \rho$*.

235. We may further confirm this conclusion *indirectly* by showing that any other supposition as to the relation between components and resultant is impossible. Let us

[See Preface.]

first assume that a supposition which we wish thus to test agrees with the foregoing so far as regards the direction of R, and only makes the length of R exceed or fall short of the diagonal D. Let us suppose then that the first resultant R_1 obtained from a and b is greater than the diagonal D_1 of the parallelogram obtained from a and b with the included angle ϕ, i.e. that $R_1 = p \,.\, D_1$, where p is an improper fraction. Now if we combine this R_1 with the force a turned in the opposite direction, the angle between the two being $\pi - \phi$ *, the new resultant R_2 deduced from them according to the same supposition must be greater than the diagonal got from R_1 and a with this same angle, still greater therefore than the other diagonal D_2 which would be got by combining D_1 which is less than R_1 and a at the same angle $\pi = \phi$ *. But we know upon purely geometrical grounds, which are quite independent of all mechanical assumptions, that this diagonal D_2 is nothing else than the given force b; R_2 then would be greater than b, whereas we know for the reasons lately stated that it must be equal to b. If now once more we compound R_2 with the given a at the angle ϕ, the resultant R_3 which would be thus obtained must for the same reasons be equal to R_1; but by the present supposition it would for the angle ϕ be equal to p times the diagonal got from R_2 and a at this angle; as then R_2 is greater than b, this diagonal also is greater than the diagonal D_1 got from a and b at the same angle; supposing it to be equal to $q\,D_1$ we get $R_3 = q\,p \,.\, D_1$, i.e. R_3 is q times as great as R_1 was. Thus the supposition that the resultant is greater than the diagonal leads to the absurd conclusion that it becomes greater and greater every time that we repeat this manœuvre in its calculation. The other supposition that it is smaller than the diagonal, i.e. that p and q are vulgar fractions, would lead to an equally impossible diminution. In order to make this indirect proof complete it would be necessary to show further that the supposition

* [$\pi - \phi$ obviously should be $\pi - \phi +$ the angle between R_1 and $\mathfrak{G}$.]

of a resultant of the same length as the diagonal but making different angles with the given forces would involve a similar absurdity, viz. that its course would be more and more deflected the oftener its calculation was repeated; and lastly it would be indispensable to prove that there is no combination of these suppositions in which the false consequences of the one would be counteracted by those of the other. But as the matter stands it is enough to state what the requirements of logic would be; we may spare ourselves the trouble of carrying them out at length.

236. Operations of synthesis or combination may always be carried out to some end, viz. to the result obtained in each case; but operations of analysis on the other hand presuppose an end which we desire to reach, though it is yet uncertain whether the subject we are treating is produced by a combination which makes this reverse process of analysis possible. Even in pure mathematics therefore the inverse operations lead to difficulties from which the direct are free; and similar doubts are suggested by the common practice of *resolving given* forces into components, though if the components were given no doubt would be felt about combining them. As any force may be split up into countless pairs of components, how, it may be asked, are we entitled to expect that any division which we arbitrarily choose will have a real validity in the complex tissue of facts present in the problem before us? In general terms this doubt is easily removed. For when we are making such a resolution in practice we always put one of the components in a direction in which some resistance or some counteracting force is foreseen or known to be present; we only resolve therefore for convenience in formulating our calculation; what we really do is to compound; if we combine the given counterforces or resistances W with the given force F, the resultant thus got is identical with that which would be obtained from the uncancelled remainder of the one component of F and the whole of the other component

which would meet with no resistance. But a real difficulty arises when the direction of the resistance itself is not immediately given and an attempt is made—in a manner that seems to me hardly convincing—to arrive by an application of the law of resolution at the principle itself which is here to be followed. I allude to the supposition that a plane resists in the direction of its normal only the imparting to it of a motion which makes with it any angle ϕ. It is quite easy to see that this motion *may* be decomposed into two, of which one parallel with the plane meets no resistance because it does not act upon the plane at all, while the other perpendicular to the plane is annihilated by the resistance of the plane, or at any rate is resisted by it. But how little right we have to carry out this decomposition here as one allowed by the nature of the case will appear from the following considerations.

Let the moving body be a perfectly smooth ball, and let it move at an angle ϕ against a perfectly smooth plane E which offers an absolute resistance; contact then will take place only in the geometrical point p, to which we must ascribe the same power of absolute resistance as to all the other points of E, however this may be brought about. Now what all these other points of E have to do with the result which follows, it is impossible to imagine; we think of them indeed when we speak of the plane E; but as they are not in contact, they cannot directly contribute anything to the resistance, and in deducing the result we may set them entirely aside without altering the conditions on which the result is to depend. But if we do this and retain the point p alone, the proposition about the resistance being at right angles becomes impossible, because it becomes meaningless; for to the point p either no line is normal or any line drawn from it in any direction is normal. But another principle seems evidently to apply here: surely p, *if* it resists, will resist in the direction from which comes the motion to be resisted: there is in the first instance no conceivable

reason for action in any other direction. If then in our example p were perfectly fixed, and if at the moment of contact the line l drawn through the point p parallel to the direction of the motion did not pass through the centre of the ball, p would entirely annihilate the motion of that thread of the mass which lies in this line l; then for the rest of the mass of the ball, whose motion would not thus be annihilated, there would arise a movement of rotation, which would cause it to turn about the point p. The inference that the resistance must occur in the direction of the motion cannot moreover be obviated by conceiving the moving body to be prismatic in shape, say a cube, of which one side remains parallel to the plane E while the direction of its motion makes with E the angle ϕ. It is true that in that case two planes are brought into contact; but even now every point of that part of E which is in contact will only be able to resist the point of the cube's side which it touches in accordance with the foregoing principle, i. e. in the direction ϕ; before we could say that it would not be so we should have to prove that the presence of the adjacent points $q\ r\ s$ of the plane E helps to determine the direction of the resistance offered by the point p: only this could render possible in fact that co-operation of the plane which we have hitherto spoken of, though we have not made use of it in deducing the result.

And now surely it is clear that we shall never succeed in proving this so long as we regard E as a geometrical plane without physical mass and yet with power to offer resistance. It is not even enough to regard E as the limiting surface of an inert mass; we are obliged to add a physical hypothesis about the forces with which the mass resists encroachment upon the space it occupies. We must give the plane E some thickness therefore; contact will not take place at one point merely, but the moving body will in fact either penetrate to a certain depth and then be thrust back by the resistance of other displaced points of the mass, or without

coming into contact while it is still at a distance it will be affected by the repulsive forces of the masses united in E. And then we should have to prove with regard to these forces of all the points of the mass that in all the other directions they annihilate one another, but in the direction of the normal to the limiting surface are added to one another and combine to make the resistance which annihilates that component of the body's motion which lies in this normal but in the contrary direction. And indeed it is not at all surprising that we should be obliged to come back to an assumption of this kind: motion altogether can only take place in a real thing, not in a point or a line; still less can we hope to calculate resistances without taking count of that which is alone able to resist, viz. the physical forces of actual bodies; surfaces as surfaces and lines as lines always cut one another without any resistance at all.

[1] **238.** I will add one more mathematical example to illustrate our general directions about method. The *Taylorian* theorem attempts to determine the value $F(x+h)$ which $F x$, a function of x, assumes when the variable quantity x increases from the limiting value which it had in Fx to the new value $x+h$. To make the statement as simple as possible I will subject the problem to certain limitations: it would take us far too long to enquire here whether they are superfluous or not. I conceive Fx to be given in the shape of an analytical expression which indicates the mathematical operations or relations from which for every definite value of x flow definite values of Fx; I assume that these values of Fx remain finite for every value of x from 0 to $x+h$, and that they increase continuously as x increases continuously between these limits. In propounding the problem in this form, as one capable of a universal solution, we directly assume that the growth of

[1] § 237, which followed here, is suppressed by desire of the author as being altogether wrong ('wegen völligen Irrthums durch den Verfasser unterdrückt'). See Editor's Preface, and Appendix.

the function from its value Fx to its new value $F(x+h)$ will follow precisely the same law which the former value Fx itself followed as x grew from o to its former limiting value x, and further that this sameness of the generating law will hold good for each infinitely small increment dh by which the function now increases precisely as for each infinitely small dx by which it formerly increased. From this it follows that it must be possible to express either value of the function, and in the first instance to express Fx, as the sum of an infinite series, each member of which indicates the increase which takes place as x increases by the addition of each successive dx. Now if it were the nature of Fx that for every smallest increase of x, i.e. for every dx, it increased by the same constant quantity $m \cdot dx$, its total value at the end would be the sum of an infinite series of similar members of the form $m\,dx$: the number of these members would be just as infinite as the number of dx into which we conceive the final value of x to be divided, or by the accumulation of which we conceive it to be formed; the sum of the series is the integral $\int m\,dx = m\,x$. If on the other hand the increase of Fx for every dx depends upon the value which the growing x has already attained at the time when this dx is added, then, if the formula we are seeking is to hold good for every finite x and h, the series we now have to take must consist of nothing but similarly constructed functions of x, relative successively to the continuously increasing values of x; if we call this function fx or $f^1 x$, then $Fx = \int f^1 \cdot dx$. Now there is no reason why we should not repeat with regard to $f^1 x$ the same considerations which we have already applied to Fx; if x in $f^1 x$ now denotes a definite value out of the many values which x may assume, $f^1 x$ may also be conceived as the sum of a series whose infinitely numerous and similarly constructed members give the increments by which as each dx was added $f^1 x$ grew to its limiting value corresponding to that value of x; and so we get $f^1 x = \int f^2 x \cdot dx$, and generally

$f^m x = \int f^{m+1} x \,.\, dx$. How to obtain from a given function Fx these derivative functions of various grades, $f^1 x, f^2 x, f^m x$, and how to work back from the latter to the former, we may assume to be well known to all who are acquainted with the infinitesimal calculus.

239. These preliminary remarks really contain the solution of the problem; nevertheless I will proceed to trace it back to the following simple train of thought which may serve at the same time to illustrate another logical method.

1. Evidently $F(x+h)$ is equal to the sum of its former value Fx and the positive or negative increment R_1 which Fx has received in consequence of the growth of the variable x from x to $x+h$. In order to determine the value of R_1 we make the simplest supposition, viz. that for each of these increments dh whose aggregate amounts to h, Fx increases by the same quantity $m_1\, dh$; then $m_1 \int dh$ which is equal to $m_1 \,.\, h$ is the value of R_1, or is the total increase of Fx. This m_1 is not incalculable. For if, as we throughout assume, the increase of Fx is to depend solely upon the nature of this function, its given value Fx must have originated in the same manner in which its further growth is now to take place; i.e. while x was passing through all values from 0 to x the function then in course of formation must have exhibited for each dx the same increase which the function thus formed now exhibits for each dh, for dx differs from dh in name only. Now Fx may be universally described as the sum of a continuous series, whose general term is represented by $f^1 x \,.\, dx$ and its last term by the same expression if x stands for the definite limiting value which the variable x attains in Fx. For each dx this series increases by $f^1 x \,.\, dx$; this quantity $f^1 x$ must be constant and be equal to m_1 if the growth of Fx up to its given limiting value is assumed to have taken place in the same way as the growth from this point up to $F(x+h)$. For every dh therefore Fx increases by $f^1 x \,.\, dh$, and the sum or the integral of these elementary increments, viz.

$h \cdot f^1 x$, is the required value of R_1. The supposition here made that $f^1 x$ is constant and equal to m_1 may not hold: but as the general formula must include the cases in which it does hold good, this second term which we have found may be accepted as an abiding element of it.

2. Even if this first supposition does not hold yet $F(x+h)$ is always equal to $Fx + h \cdot f^1 x + R_2$, if we understand by R_2 the positive or negative supplement still necessary for the complete measurement of the true value of the function. As this further addition can only be required because Fx does not increase by the same amount for every dh or dx, i.e. because $f^1 x$ is no constant quantity, but dependent upon the value which the variable x has attained at each stage, it is plain that $f^1 x$ in the second term, $hf^1 x\ (= R_1)$, of our formula still denotes only the fixed particular value which the general function $f^1 x$ now to be conceived as variable, assumes when the variable x assumes its limiting value x or when the variable h is equal to 0. We cannot therefore retain this second term $h \cdot f^1 x$ unless to each of the terms $f^1 x \cdot dh$ of which it is the sum we add the further increase exhibited by the limiting value of $f^1 x$ contained therein for each increment dh of the variable h. For this increase again we make the simplest supposition, viz. that it is the same for each dh and is equal to $m_2\, dh$. This m_2 is also capable of determination. For once more if our supposition is to hold good it must react upon Fx also; the same law by which this function is now to increase must have regulated its origin; the increase of $f^1 x$ must have been the same for each dx and equal to $m_2\, dx$. Now $f^1 x$ is the sum of a continuous series whose general term is $f^2 x \cdot dx$; this then is the very increment by which this series or its sum $f^1 x$ continuously increases each time that x is increased by dx; our condition is fulfilled therefore if we put down $f^2 x$ as constant and equal to m_2: then the growth of Fx beyond its given value follows the same law which regulated its formation up

to that point. Its total increase therefore is the sum of two series; the first of these consists entirely of similar terms $f^1 x \,.\, d\,h$, and its sum $= R_1$; the second represented by R_2 contains increasing terms, the first term $f^2 x \,.\, d\,h$ represents the first new increase which $F x$ exhibits when the former limiting value x of the variable x is increased by the first $d\,h$, or when the variable h, growing from 0, attains its first value $d\,h$; each successive $(n+1)^{\text{th}}$ term is formed by adding the same increment $f^2 x \,.\, d\,h$ to the value of the n^{th} term; $h \,.\, f^2 x \,.\, d\,h$ therefore is the general term of this second series, and is what we must add as supplement to the general term of the first series. The total increase of $F x$ is therefore the sum of the continuous series $(f^1 x + h f^2 x) d\,h$, or $h \,.\, f^1 x + \dfrac{h^2}{1 \,.\, 2} \cdot f^2 x$; the second term of this expression is the required value of R_2.

3. If a given function $F x$ were of such a nature that even this second supposition was not enough to exhaust its growth, we should still be always able to retain the terms of the formula already found if we added a fresh R_3 to supplement them. And to determine this R_3 we should repeat the same process as before. We could only require it because $f^2 x$ also is not constant, but is dependent upon the value which x has attained at any point and increases with it. Let us assume that these increments are at least constant for each $d\,h$ and equal to $m_3\, d\,h$. If then we express $f^2 x$ as the sum of a continuous series whose general term is $f^3 x \,.\, d\,x$, we have but to put down $f^3 x$ as constant and equal to m_3, and we thereby make sure that our general condition is satisfied and that $F x$ has grown to this its given limiting value in the same way as it is now to grow beyond it. Now R_2, the third term of our formula, was the sum of a continuous series, whose general term is $h \,.\, f^2 x \,.\, d\,h$; if then we form a second series containing the additions by which R_2 is to be supplemented, $h \,.\, f^3 x \,.\, d\,h$ will be the amount by which each $(n+1)^{\text{th}}$ term of this

series exceeds the n^{th} term; $\int h.f^3 x\, dh$ therefore or $\frac{h^2}{1 . 2} \cdot f^2 x$ is the general term of this series R_3. We obtain the second and third increment of Fx therefore by summing the continuous series whose general term is now

$$\left[hf^2x + \frac{h^2}{1 . 2} \cdot f^3 x\right] dh,$$

and the result is that

$$R_2 + R_3 = \frac{h^2}{1 . 2} \cdot f^2 x + \frac{h^3}{1 . 2 . 3} \cdot f^3 x.$$

4. It would be useless to carry this process further; it will readily be seen that if we constantly repeat the assumptions here made the required formula will assume the familiar shape of the Taylorian series, viz.

$$F(x+h) = Fx + \frac{h}{1} \cdot f^1 x + \frac{h^2}{1 . 2} \cdot f^2 x + \frac{h^3}{1 . 2 . 3} \cdot f^3 x \ldots\ldots + \frac{h^m}{1 . 2 . 3 \ldots m} \cdot f^m x + R_{m+1}.$$

But this formula would be of little value if the very assumptions on which it rests could not be shown to be the only admissible assumptions. It would be beyond all doubt logically correct, but only in the sense in which the barrenest of tautologies is correct, if it only meant that any quantity M might always be expressed by a series of quite arbitrary terms provided that we reserved the right to add a remaining term R intended to make good all the errors which we had committed by making M equal to the series. The formula has a serviceable meaning only when we do not need this compensating remainder, i.e. when we can prove that the value of $F(x+h)$ can be completely expressed either by a finite number of the developed terms, or by a series of such terms which though infinite yet converges so as to admit of being summed. But how do we learn that this is the case? From the fact that for a given function Fx one of its derivative functions $f^m x$ turns out upon actual calculation to be equal to 0, that the series

therefore breaks off before the term which contains it, we plainly can infer nothing but that there is no further increase of $F x$ that can be got by the further development of the series we have taken; the inference that no other increase can occur at all would imply that we could prove that this very mode of calculation must include all increase of which $F x$ is by its nature capable. Now this point we think no longer needs special demonstration; it is contained in the assumption which we made that $F x$ does not increase under any other condition than that of the continuous uniform increase of x, and that its mathematical structure remains the same for every one of the values of x which have been reached. If then a function grows in such a way that for every $d h$ it exhibits the same constant increase, while at the same time every $d h$ that thus enters into it becomes the starting-point of a new constant increase, we get as the expression of its total increase through the interval h an infinite series, in whose terms the one set of factors h, $\frac{h^2}{1 \cdot 2}$, $\frac{h^m}{1 \cdot 2 \ldots m}$ depend for their form simply upon this universal form of growth and are therefore similar in form for all functions. But in order that this series may give the specific growth of each particular function in distinction from that of any other, the other set of factors $f^1 x$, $f^2 x$, $f^3 x$ are added to these universal factors in such a way that each of them indicates the particular magnitude, dependent in each case upon the nature of the given $F x$, of the first, second, third, or m^{th} increase which occurs for each $d h$; the series, as the complete expression of $F(x+h)$, closes when one of these factors vanishes. The developed terms of the series above given were therefore not arbitrarily assumed; what we meant to do with them was to measure $F(x+h)$, not by a standard foreign to the nature of this function, but by the standard supplied by the function itself and by the nature of its assumed growth; if by this standard the value of $F(x+h)$ can be expressed in a finite number

of terms or in a number which though infinite admits of being summed, there can be no increase derived from other sources which would have to be added to this. For however a function may grow,—provided only that it is subject at no stage of its growth to the introduction of new conditions from without,—the continued repetition of the assumptions above made (first of a constant increase, then of a constant positive or negative increase of this increase, then of a fresh constant positive or negative increase of this second increase, and so on) will enable us to exhaust the total value of the resulting growth just as certainly as we are enabled to express any curved path by properly chosen epicycles, or any irrational number by an infinite series of positive and negative powers of ten. Taken in this sense, as a mere definition of growth, the series remains *logically valid* even when it is rendered *mathematically useless* by divergence for a demonstrably finite increase of the function. If it were not so, then, even if it were possible to restore convergence by transforming the function without altering its content, the result it yielded could only be regarded as correct in fact, supposing it could be shown to be correct,—it could not be regarded beforehand as obviously and necessarily correct: such transformation only serves to bring within the limits of calculability what holds good as it stands.

END OF VOLUME I.